Quality Engineering, 8(1), 75–91 (1995–96)

ANALYTIC PARAMETER DESIGN

SØREN BISGAARD and BRUCE ANKENMAN
Center for Quality and Productivity Improvement
and Department of Industrial Engineering
University of Wisconsin–Madison
Madison, Wisconsin 53705

Introduction

Off-line quality control as introduced by Taguchi and Wu (1) provides quality engineers with a comprehensive systems approach to the product design process. It consists of the following three steps: *system design, parameter design,* and *tolerance design*. We will review briefly the key ideas below but refer to Ref. 2 for a more comprehensive discussion. System design is the phase of the product design process where the general layout of the prospective product is established. Parameter design, which is the next logical phase, is concerned with the optimization of the system found in the previous phase. Usually, the objective of parameter design is to minimize variation in the product's output caused by variation in components or in the environment by manipulation of the system's parameters while maintaining some specified target output. According to Taguchi, after having determined the optimal parameter settings in the parameter design phase, the designer is then asked to provide tolerances that balance the desire to reduce the variability in the output with the need to allow for variability in the input. This is the third and final step called tolerance design.

To facilitate the parameter design optimization, Taguchi and Wu (1) suggest the use of experimental design methods and, in particular, orthogonal arrays laid out as inner- and outer-array designs. Taguchi also provides a variety of examples of the use of experimental design methods for optimizing products. Those examples can be classified broadly into two groups; those for which a functional systems relation is known between the inputs and the output or performance measure of the product, and those for which no such relation is available. In the former case, Box and Fung (3) have pointed out that the use of experimental design is inefficient, and they show that better approaches based on the well-known error transmission formula are available. The latter case, called the dual-response problem, has been addressed with response surface methodology by Vining and Myers (4), Myers, Khuri, and Vining (5), and Del Castillo and Montgomery (6).

In this article, we build on Box and Fung's work for the case when a mathematical systems equation exists for the product. We extend their use of the error transmission formula and, assisted by symbolic manipulation software, we use analytic approaches to solve the parameter design problem. Specifically, we use modern methods of nonlinear programming in combination with analytically derived derivatives and have found that such an approach provides a reliable determination of the optimum conditions. As symbolic manipulation software is rapidly maturing, we expect that such an approach soon will allow us to handle larger and more complex problems. In addition, and perhaps more importantly, we will also demonstrate that analytic methods provide more insight into the problem, give a better understanding of the optimal conditions, and can supply us with useful measures of sensitivity.

We have deliberately chosen a very simple electrical circuit originally used by Taguchi (7) for our discussion of analytic parameter design. The simplicity of this example allows us to support the conceptual development with insight-generating three-dimensional graphics. As to the organization of the article, we first provide a brief literature overview followed by a section where we introduce a slightly modified version of the simple electrical circuit used by Taguchi (8). Next, we provide a section on a general formulation of the analytic parameter design problem using the concepts of nonlinear programming which clarify several subtleties of the problem. We then show how to find the optimal solution to the electrical circuit for a given set of input variances using both analytic and graphical methods. This is then followed by a discussion of the tolerance design problem where the input variances are varied. We proceed to show that the optimal solution in the parameter design stage depends on the tolerances. This, in turn, leads us to the observation that the parameter design and the tolerance design problems cannot, as implied by Taguchi and others, be considered separately. Thus, parameter design and tolerance design must either be performed simultaneously or solved through several iterations. The article is concluded with a discussion.

Background

During the past decade, parameter design has received a great deal of attention in the literature. A large number of articles about parameter design have been published, and an even larger number of papers have been presented at recent conferences. A panel discussion edited by Nair (9) summarizes much of the work in recent years.

Perhaps the earliest article on parameter design is by Morrison (10) and appeared long before the term parameter design was coined. He discusses how the variation in several manufacturing dimensions affects the variation in the volume of glass beads and rings used in vacuum envelopes and glass-to-metal seals. In this article, he studies several product designs and shows how to determine which manufacturing dimensions are important to control. Morrison (10) also suggests an analysis, based on the error transmission formula, which can "guide the designer towards better proportions or a better arrangement in the design, so as to reduce overall variance." This important but much neglected article provides an early contribution to robust parameter design.

Our literature search then shows that between the publication of Morrison's article and the path-breaking series of publications by Taguchi, which began to appear in the early 1980s, little if any was published on the subject. In response to Taguchi's articles, Box and Fung (3) then demonstrated a more efficient method of parameter design using numerical derivatives and numerical methods of nonlinear programming when a functional relationship for the system is known. They also discuss, in the context of Taguchi and Wu's (1) Wheatstone bridge example, how sensitive the optimum design values are to changes in the assumptions about the component variances.

Lately, a number of conference papers have appeared in the engineering literature which have suggested using various optimization methods for parameter design. Wilde (11), for example, considers the same electrical circuit example that we use. He puts the problem into a nonlinear programming form but does not use the error transmission formula as the basis for variance reduction. Hsieh, Oh, and Oh (12) use the optimizer IDESIGN (13) to determine the optimal design for minimizing the error associated with a specific manufacturing technique called *net building*. Net building is when a particular output of the product is adjusted to target as the final step on the production line. Parkinson et al. (14) provide a general discussion of a two-step method as an alternative to standard nonlinear programming techniques. They use a two-bar truss as an example of a design problem. The technique they use involves optimizing a design without taking into account the variation in the constraints. This gives what they call the nominal optimum. In the second step, the derivatives of the constraints evaluated at the nominal optimum are used to estimate the constraint variance. The problem is then reoptimized using these estimates.

Sundaresan, Ishii, and Houser (15) use Taguchi's orthogonal-array approach to optimize a gear design. Michelena and Agogino (16), like Wilde (11), use Taguchi's electrical circuit example. They expand on Wilde's work and come to a slightly different solution. They also provide a comparison of their solution with Wilde's and Taguchi's solutions. Ramakrishnan and Rao (17) emphasize the use of Taguchi's loss function in conjunction with nonlinear optimization to minimize variation in machining and welding processes. Box and Fung (22) expand on their earlier work on parameter design and stress the ironic fact that the robust solution is not robust to the assumptions made about the component variances. Several other articles are pertinent to this area, but we have selected the earliest and the latest of these for this review. For further details on the literature, see Ref. 9.

Taguchi's *RL* Circuit Example

As indicated earlier, we will use a simple alternating current circuit (see Fig. 1) as our leading example of a "design." The "product," which we will refer to as an *RL* circuit, consists of a voltage source, a resistor *R*, and an inductor *L* connected in series. In itself, as a design problem, this circuit is, of course, trivial. However, as with the inclined plane in mechanical physics, its simplicity allows us to demonstrate the fundamental principles of parameter design, of nonlinear optimization as applied to the parameter design problem, and to do so with graphics. The understanding gained from this simple example will provide a conceptual guide for how to deal with more complex design problems.

According to Taguchi the design objective for the *RL* circuit is to set the nominal values of resistance and inductance such that the average root mean squared (RMS) current is targeted at 10 A, and the variation around that target is minimized. (Apparently, the phase change is not important.) From alternating-current circuit theory, it is then established readily that

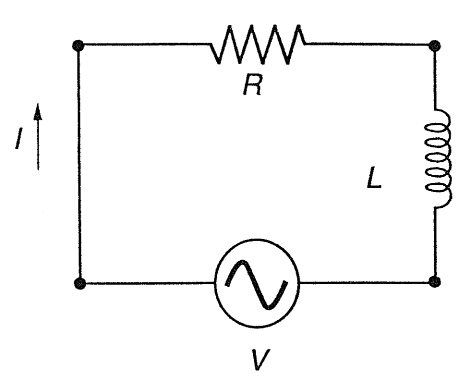

Figure 1. The *RL* circuit.

$$I = \frac{V}{\sqrt{(2\pi f L)^2 + R^2}} \tag{1}$$

where R is the resistance, L is the inductance, V is the voltage, f is the frequency, and I is the current. We will call R and L the *design variables*, as their nominal values can be controlled by the designer. On the other hand, V and f are factors that are not under the control of the designer. Thus, we refer to those factors as *environmental variables*. In our example, there are two design variables, two environmental variables, and one target value for the response. Hence, it can readily be seen that there will be a number of solutions in R and L that will satisfy the target of 10 A of current. Moreover, we note that in volume production, the resistance and inductance values will vary randomly from component to component, and the voltage and frequency will vary both from installation to installation as well as over time. As a consequence of these random variations, there will be variation in the current. The problem is how to minimize the variation in the current I. To achieve that, the designer may either reduce the manufacturing tolerances of design variables, in this case variance of the resistor and the inductor, or reduce the sensitivity of the current to variation in both the design and environmental variables. The latter approach is parameter design.

Mathematical Formulation of the Parameter Design Problem

As already indicated, the purpose of this article is to provide a general approach to the parameter design problem and to bring out some of this problem's subtleties. We will, therefore, now reformulate the simple *RL* circuit parameter design problem described in the previous section using the terminology and symbolism commonly used in the nonlinear optimization literature. Although we will provide some of the important concepts of nonlinear optimization in the following discussion, we refer to Ref. 18 for more details on this topic.

Now turning to a general formulation of our robust circuit problem, let the functional relationship (1) be denoted by $g(\bullet)$, and let $\mathbf{x}$ denote an n-dimensional vector of nominal values of the design variables and $\mathbf{z}$ an m-dimensional vector of assumed nominal values of the environmental variables. As indicated, in applications the design and environmental variables will be perturbed by random errors. Thus, let $\mathbf{X} = \mathbf{x} + \varepsilon_x$ and $\mathbf{Z} = \mathbf{z} + \varepsilon_z$ where we will assume for simplicity that the errors ε_x and ε_z are all mutually independent and that $E\{\varepsilon_x\} = 0$ and $E\{\varepsilon_z\}=0$. Now let the (random) output characteristic, in our case the current, be denoted by Y. Thus, with this notation, we may write (1) as $Y = g(\mathbf{X}, \mathbf{Z})$.

Now suppose the designer wants to target the expected value of the response $E\{Y\}$ at some target, T, with minimum variation. To achieve that, the designer may control the nominal value of the design parameter vector $\mathbf{x} = E\{\mathbf{X}\}$. We may then write the parameter design problem as

$$
\begin{aligned}
&\min_{\mathbf{x}} V\{g(\mathbf{X}, \mathbf{Z})\} \\
&\text{subject to} \quad E\{g(\mathbf{X}, \mathbf{Z})\} = T, \\
&\qquad\qquad\qquad E\{\mathbf{X}\} \in \mathbf{Q}^n \subseteq \mathbf{R}^n, \\
&\qquad\qquad\qquad E\{\mathbf{Z}\} = \mathbf{z}, \\
&\qquad\qquad \textit{and} \quad V\{Z\} = \boldsymbol{\sigma}_{\mathbf{z}}^2,
\end{aligned}
\tag{2}
$$

where $\mathbf{Q}^n$ is the feasible design set, $\mathbf{R}^n$ is the n-space of real numbers, and $\mathbf{z}$ and $\boldsymbol{\sigma}_{\mathbf{z}}^2$ are m-vectors of the assumed values of the means and variances of the environmental variables, respectively.

There are many technical intricacies in this minimization problem that could be the subject of a larger discussion. Some of these are related to the distribution of the response function, the interdependence between and among the $\mathbf{X}$ and $\mathbf{Z}$ variables, the relationship between the mean and standard deviation of the individual random variables, and the definition of the feasible design set. However, our purpose is to outline a general framework for parameter and tolerance design in the context of nonlinear optimization and to draw broad conclusions about the parameter design problem. To avoid getting bogged down in details, we will, therefore, assume that each of the individual elements of the vectors $\mathbf{X}$ and $\mathbf{Z}$ are mutually independent random variables. Thus, we will assume that $V\{\mathbf{X}\} = \Sigma_{\mathbf{x}} = \text{diag}\{\sigma_{x_i}^2\}$ and $V\{\mathbf{Z}\} = \Sigma_{\mathbf{z}} = \text{diag}\{\sigma_{z_j}^2\}$, where $\sigma_{x_i}^2 = V\{X_i\}$, $i = 1, \ldots, n$ and $\sigma_{z_j}^2 = V\{Z_j\}$, $j = 1, \ldots, m$. In some applications, it will further be important to allow the variances of the design variables to be functions of their means. When that is the case, we may write $\sigma_{x_i}^2 = \sigma_{x_i}^2(x_i)$, i $= 1, \ldots, n$.

To make some inroads on the general but rather intractable problem as stated in Eq. (2), let us now proceed to approximate the variance and the expected value of the response. Specifically, we will use the standard first-order error transmission formula,

$$
\begin{aligned}
V\{g(\mathbf{X}, \mathbf{Z})\} &\approx \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{m} \left(\frac{\partial g}{\partial z_i} \right)^2 \sigma_{z_i}^2 \\
&= f(\mathbf{x}, \mathbf{z}, \boldsymbol{\sigma}_{\mathbf{x}}, \boldsymbol{\sigma}_{\mathbf{z}})
\end{aligned}
\tag{3}
$$

to approximate the variance. We will also use the approximation

$$E\{g(\mathbf{X}, \mathbf{Z})\} \approx g(E\{\mathbf{X}\}, E\{\mathbf{Z}\}) = g(\mathbf{x}, \mathbf{z}).$$

According to Ku (19), these approximations provide surprisingly good estimates of the transmitted variance when the variables are independent and are subject to some relatively mild conditions discussed in detail in his section 3.5. Fathi (20) provides a more accurate but more complicated formulation of this problem based on the second-order Taylor approximation to the variance.

To further simplify the general formulation in Eq. (2), let $h(\mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{z}) - T$. With these simplifications, we may now write the parameter design problem as a deterministic nonlinear optimization problem

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}, \boldsymbol{\sigma}_{\mathbf{x}}, \boldsymbol{\sigma}_{\mathbf{z}}) \\ &\textit{subject to} \quad h(\mathbf{x}, \mathbf{z}) = 0 \\ &\mathbf{x} \in \mathbf{Q}^n \subseteq \mathbf{R}^n, \end{aligned} \tag{4}$$

where $\mathbf{z} = E\{\mathbf{Z}\}$ and $\boldsymbol{\sigma}_{\mathbf{z}}^2 = V\{\mathbf{Z}\}$.

In this form, the problem can be solved using standard optimization techniques to obtain the optimal settings $\mathbf{x}^o$. In addition, sensitivity measures can also be calculated at $(\mathbf{x}^o, \mathbf{z})$ to determine how robust the optimum is to changes in the assumptions of the problem formulation. In particular, if the Lagrange multiplier method is used, the multiplier provides a sensitivity measure that estimates the potential reduction in the response variance as a function of a change of the target value, T. We will discuss this below.

Solution to the *RL* Circuit

Let us now return to Taguchi's *RL* circuit example. The problem is to set the mean values of the resistance R and the inductance L such that the expected value of the output I is 10 A with minimum variation. We will use notation consistent with the general formulation above and use example values roughly consistent with those by Taguchi (8). The current will be denoted by $g(\mathbf{X}, \mathbf{Z})$ because it is the output of the circuit. It depends on the random design variables, resistance X_R and inductance X_L, and the random environmental variables, voltage Z_V and frequency Z_f. The mean values for the environmental variables are fixed and assumed to be $z_f = 60$ Hz and $z_V = 100$ V. The mean values of the design variables are under the control of the designer and denoted x_R and x_L. The standard deviations of the environmental variables are assumed to be $\sigma_V = 8.16$ for the voltage and $\sigma_f = 0.1$ for the frequency. We point out that we have assumed a smaller and more realistic standard deviation for the frequency than Taguchi did, a fact that will prevent us from a comparison of his solution with ours.

In Taguchi's formulation of the problem the standard deviations of the design variables are proportional to their mean levels, so we write

$$\sigma_i^2 = \sigma_i^2(x_i) = (\phi_i x_i)^2, \quad i = R,\ L, \tag{5}$$

where the ϕ_i's are proportionality constants. For this case, the ϕ_i's are assumed to be equal such that $\phi_R = \phi_L = 0.08$. Thus, the standard deviation of the resistance is $\sigma_R = 0.08x_R$ and the standard deviation of the inductance is $\sigma_L = 0.08x_L$.

The response function (1) can now be rewritten as

$$g(\mathbf{X}, \mathbf{Z}) = Z_V[(2\pi Z_f Z_L)^2 + X_R^2]^{-1/2}. \tag{6}$$

Thus, $h(\mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{z}) - T = z_V[(2\pi z_f x_L)^2 + x_R^2]^{-1/2} - T$. Using Eqs. (3) and (5), we have

$$f(\mathbf{x}, \mathbf{z}, \boldsymbol{\phi}, \boldsymbol{\sigma}_z) = \left(\frac{\partial g}{\partial x_R}\right)^2 \phi_R^2 x_R^2 + \left(\frac{\partial g}{\partial x_L}\right)^2 \phi_L^2 x_L^2 + \left(\frac{\partial g}{\partial z_V}\right)^2 \sigma_V^2 + \left(\frac{\partial g}{\partial z_f}\right)^2 \sigma_f^2, \tag{7}$$

where

$$\frac{\partial g}{\partial x_R} = -x_R z_V[(2\pi z_f x_L)^2 + x_R^2]^{-3/2},$$

$$\frac{\partial g}{\partial x_L} = -(2\pi z_f)^2 x_L z_V[(2\pi z_f x_L)^2 + x_R^2]^{-3/2},$$

$$\frac{\partial g}{\partial z_V} = [(2\pi z_f x_L)^2 + x_R^2]^{-1/2},$$

and

$$\frac{\partial g}{\partial z_f} = -(2\pi x_L)^2 z_f z_V[(2\pi z_f x_L)^2 + x_R^2]^{-3/2}.$$

We note in passing that these derivatives can be found easily using a symbolic manipulation software package such as *Mathematica, Maple*, or *Macsyma*. Now inserting the numerical values given above, the parameter problem can be expressed as a constrained minimization problem:

$$\begin{aligned} &\min_{\mathbf{x}}\{f(\mathbf{x}) = (64.0x_R^4 + 1.33 \times 10^{10}\pi^4 x_L^4)[(120\pi x_L)^2 + x_R^2]^{-3} \\ &\qquad\qquad + 66.6[(120\pi x_L)^2 + x_R^2]^{-1}\} \\ &\textit{subject to } h(\mathbf{x}) = 100[(120\pi x_L)^2 + x_R^2]^{-1/2} - 10.0 = 0 \\ &\textit{and } \mathbf{x} \in \mathbf{Q}^2 \subseteq \mathbf{R}^2. \end{aligned} \tag{8}$$

To provide a better understanding of the optimization, let us now discuss a geometric solution to this problem. First, note that x_R and x_L must be non-

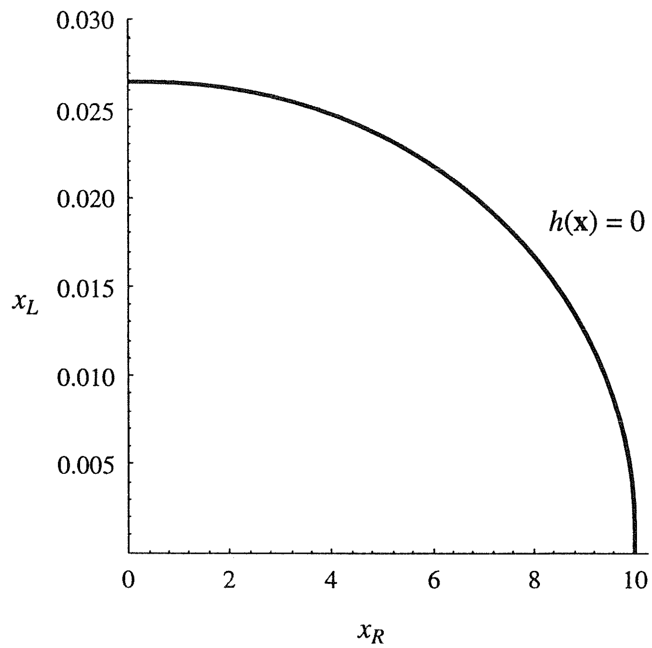

Figure 2. The feasible region for the constraint $h(\mathbf{x}) = 0$.

negative. Second, the feasible set consists of the points given by $h(\mathbf{x}) = 0$. The feasible set as a subset of the space spanned by x_R and x_L is shown in Figure 2.

Next, let us consider a three-dimensional plot of the variance surface, $f(\mathbf{x})$, as a function of x_R and x_L. Figure 3 shows the surface $f(\mathbf{x})$ superimposed with the feasible curve, $h(\mathbf{x}) = 0$. The intersection between these two surfaces is the set of feasible solutions and the corresponding value of the variance of the current. The lowest point of the variance on the feasible set is indicated by $f(\mathbf{x}^o)$; thus, the point $\mathbf{x}^o = (x_R^o, x_L^o)'$ provides the optimal value for the constrained optimization problem.

Lagrange Methods and Secondary Conditions

The graphical approach used in the previous section provides useful conceptual insights into the problem of constrained optimization relative to parameter design problems. However, a geometric approach is limited clearly to three dimensions; other methods are necessary for more realistic problems. In this section, we will, therefore, discuss a general method for solving constrained optimization problems using Lagrange multipliers which will be useful for parameter design problems of larger dimensionality. We acknowledge that in the above problem it would be easy to solve the constraining equation explicitly in one of the two variables, substitute this expression into the objective

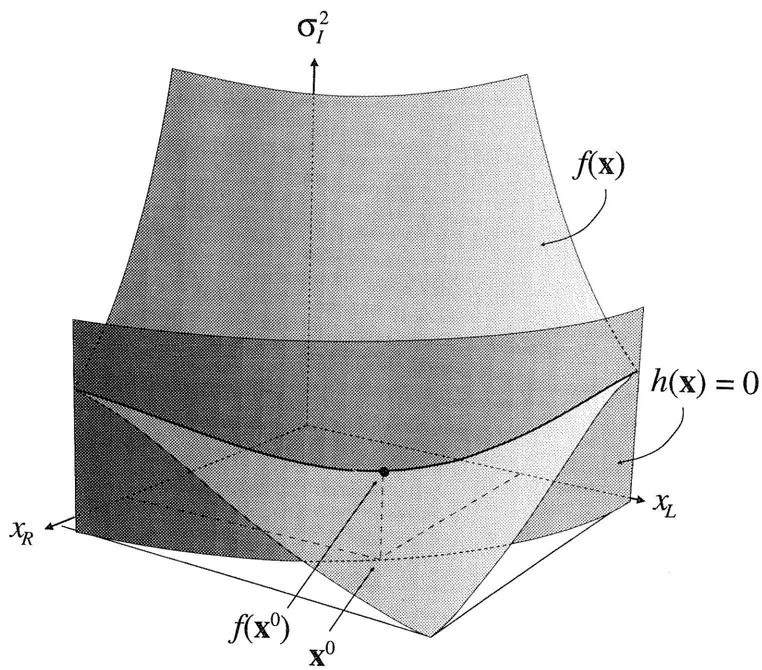

Figure 3. The graphical solution, $\mathbf{x}^o = (x_R^o, x_L^o)$, of the parameter design problem. The solution is found by locating $f(\mathbf{x}^o)$, the lowest point on the variance surface $f(\mathbf{x})$, which is also in the feasible region for the constraint $h(\mathbf{x}) = 0$.

function, and solve the optimization problem directly. However, we prefer, here, for demonstration purposes the more general approach using Lagrange multipliers.

The first step in this approach is to convert the constrained optimization problem to an unconstrained problem by using the Lagrangian function $L(\mathbf{x}, \lambda)$, defined as

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda h(\mathbf{x}), \tag{9}$$

where λ is an undetermined multiplier. The Lagrangian function has the property that a minimum of the unconstrained function $L(\mathbf{x}, \lambda)$ is also a solution of the constrained problem (8). For a particularly lucid discussion of this, see Ref. 21, p. 192). Thus, we may use unconstrained optimization methods to minimize $L(\mathbf{x}, \lambda)$ and thereby obtain a solution to the *RL* circuit problem.

Necessary conditions for a stationary point of $L(\mathbf{x}, \lambda)$ at the point $(\mathbf{x}^o, \lambda^o)$ are

$$\nabla_{\mathbf{x}} L(\mathbf{x}^o, \lambda^o) = \nabla_{\mathbf{x}} f(\mathbf{x}^o) - \lambda^o \nabla_x h(\mathbf{x}^o) = \mathbf{0} \tag{10}$$

and

$$\left.\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda}\right|_{\substack{\mathbf{x}=\mathbf{x}^o \\ \lambda=\lambda^o}} = -h(\mathbf{x}^o) = 0. \tag{11}$$

For Eq. (10) to hold, the two gradient vectors $\nabla_{\mathbf{x}} f(\mathbf{x}^o)$ and $\nabla_{\mathbf{x}} h(\mathbf{x}^o)$ must be proportional, and hence parallel, with proportionality factor λ^o. Below, we will explore the significance of this. The second condition, Eq. (11), is merely a restatement of the constraint $h(\mathbf{x}) = 0$ at the optimal point, $\mathbf{x}^o$.

Note that the Lagrange multiplier λ^o has a practical interpretation. At the stationary point,

$$\lambda^o \nabla_{\mathbf{x}} h(\mathbf{x}^o) = \nabla_{\mathbf{x}} f(\mathbf{x}^o),$$

which means that the Lagrange multiplier provides an index for how much the objective function $f(\mathbf{x})$ will change relative to a change on the right-hand side of the constraint $h(\mathbf{x}) = 0$. The Lagrange multiplier is therefore, in economics and operations research, often referred to as the *shadow price*. Thus, if the objective function is expressed in economic terms, the Lagrange multiplier shows how much we should be willing to pay for a unit change of the constraint. For further discussion of the general theory of Lagrange multipliers, see Ref. 18.

Let us now return to our *RL* circuit. The solution to Eqs. (10) and (11) are $\mathbf{x}^o = (x_R^o, x_L^o)' = (7.072, 0.0188)'$ and $\lambda^o = 0.1972$. At $\mathbf{x}^o$, the objective function is $f(\mathbf{x}^o) = 0.985$ and the two gradient vectors are $\nabla_{\mathbf{x}} h(\mathbf{x}^o) = (-0.7072, -0.266.5)'$ and $\nabla_{\mathbf{x}} f(\mathbf{x}^o) = (-0.1395, -52.56)'$. Thus, we see that they are parallel with proportionality constant $\lambda^o = 0.1972$.

For our circuit example, with only two dimensions, we have found our solutions through analytic methods both with respect to the differentiation for the variance approximation (3) and for the necessary differentiation involved in finding the solution to Eq. (4) using Eqs. (9)–(11). In most practical problems, however, numerical methods of some kind will be needed. Now it is well recognized that numerical differentiation is often unstable. In the light of the rapid progress in symbolic manipulation software, we, therefore, recommend that whenever possible, analytic derivatives be used and evaluated. It is especially recommended that the differentiation involved in the variance approximation (3) is done analytically. This will more likely assure convergence to a proper solution. There are now many standard software packages available for finding the solution of nonlinear optimization problems using Lagrange multipliers and other techniques, once the problem is precisely stated. However, as our focus in this article has been on providing a better conceptual understanding of the parameter design problem and its geometry, we have chosen not to dwell too much on the numerical calculations involved.

So far, we have discussed first-order or necessary conditions for a stationary point to be optimal for the Lagrangian. A thorough analysis, however, should also include a discussion of the second-order conditions to assure that the stationary point is indeed a minimum. For a general nonlinear optimization problem, a set of sufficient conditions for an optimal solution is provided by the Kuhn–Tucker conditions (see Ref. 18, p. 314). However, the parameter design problem, as we have discussed it, only involves differentiable equality constraints. Thus, finding sufficient conditions for an interior stationary point (not located on the boundary of the feasible set) to be a minimum reduces to an analysis of the Hessian. Specifically, we need to verify that the Hessian of the Lagrangian is positive definite at $\mathbf{x}^o$.

The stationary point $\mathbf{x}^o = (x_R^o, x_L^o)' = (7.072, 0.0188)'$ is clearly an interior point, so we now compute the Hessian of the Lagrangian. For our two-dimensional problem, the Hessian, $\mathbf{H}(\mathbf{x}^o, \lambda^o)$, is

$$\mathbf{H}(\mathbf{x}^o, \lambda^o) = \begin{pmatrix} \left.\dfrac{\partial^2 L(\mathbf{x}, \lambda)}{\partial^2 x_R}\right|_{\substack{\mathbf{x}=\mathbf{x}^o \\ \lambda=\lambda^o}} & \left.\dfrac{\partial^2 L(\mathbf{x}, \lambda)}{\partial x_R\, \partial x_L}\right|_{\substack{\mathbf{x}=\mathbf{x}^o \\ \lambda=\lambda^o}} \\ \left.\dfrac{\partial^2 L(\mathbf{x}, \lambda)}{\partial x_R\, \partial x_L}\right|_{\substack{\mathbf{x}=\mathbf{x}^o \\ \lambda=\lambda^o}} & \left.\dfrac{\partial^2 L(\mathbf{x}, \lambda)}{\partial^2 x_L}\right|_{\substack{\mathbf{x}=\mathbf{x}^o \\ \lambda=\lambda^o}} \end{pmatrix}, \tag{12}$$

and for the *RL* circuit, at the stationary point, it can be shown that

$$\mathbf{H}(\mathbf{x}^o, \lambda^o) = \begin{pmatrix} 0.0226 & -1.110 \\ -1.110 & 3220.87 \end{pmatrix} \tag{13}$$

which is positive definite because it has positive eigenvalues, 0.022 and 3220.87. Thus, we conclude that $\mathbf{x}^o$ is a minimum. For more details on this condition, see Ref. 18, p. 316.

With the optimum established, let us now perform a sensitivity analysis. Recall that the Lagrange multiplier is the ratio of the length of the gradient vectors $\nabla_{\mathbf{x}} f(\mathbf{x}^o)$ and $\nabla_{\mathbf{x}} h(\mathbf{x}^o)$. Hence, $\lambda^o = 0.1972$ is an estimate of the amount the objective function f will change for a given change on the right-hand side of the constraint h. It will, therefore, provide an estimate of the reduction in the response variance for a unit change in the target value. Thus, if we subtract the Lagrange multiplier from the objective function at the optimum point, we will get an estimate of how much the variance of the current would be reduced if the target current were lowered by 1 A or if $T = 9$. Doing this, we find

$$\left.\hat{f}(\mathbf{x}^{oo})\right|_{T=9} = f(\mathbf{x}^o) - \lambda^o = 0.985 - 0.1972 = 0.7878.$$

To check how accurate this linear approximation is, we re-solved the problem for $T = 9$ and found that $f(\mathbf{x}^{oo})|_{T=9} = 0.799$, which is quite close to our approximate estimate.

In many cases, the target is fixed and the information provided by the Lagrange multiplier cannot be used. However, in some cases, a high absolute value for the Lagrange multiplier might prompt the designer to explore the possibility of biasing the target value in order to reduce the output variance. In a separate investigation, which will not be reported here, we also investigated the possibility of minimizing the mean squared deviation from the target rather than the variance, subject to being on the target. However, for the examples considered, this change of objective function did not substantially change the solution.

In keeping with the findings of Box and Fung (22), we note that the solution presented above is very sensitive to the assumption that the standard deviations of the design variables are proportional to their mean levels. For this example, we will maintain that assumption, as electrical components are almost always sold with percentage tolerances. However, in the next section, we will explore how the optimum solution changes when the designer considers different percentage tolerances for the components.

Tolerance Design

In the *RL* circuit example, the parameter design problem was solved assuming fixed values for ϕ_L and ϕ_R, the constants related to the percentage variation of the resistor and the inductor. However, in practice, the designer may have control over this variation by specifying the tolerances of the components. Of course, specifying tight tolerances may increase the cost, and relaxing them may cause excessive variation in the current. In order to choose a proper balance in the tolerance design, we may consider the consequences of varying ϕ_L and ϕ_R over chosen ranges. We then repeat the constrained parameter design optimization to find the minimum attainable variation in the current, $\psi = \psi(\phi_R, \phi_L) = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \phi)$, for each combination of ϕ_L and ϕ_R. Figure 4 shows a contour plot of ψ.

Whereas the Lagrange multiplier provided a sensitivity analysis of the consequences of changing the constraints, this analysis provides a sensitivity analysis of changing the input variances. Figure 4 shows that there exists a set of parameter and tolerance specifications which give the same value of variance in the current. However, to achieve the same variance in the current with different tolerances, it is necessary to change the parameter values of resistance and inductance. For example, when $\phi_R = 0.068$ and $\phi_L = 0.1015$, the minimum attainable ψ is 0.985, as it was in the previous section. However, the optimal parameter values have now changed to $\mathbf{x}^o = (x_R^o, x_L^o)' = (8.31, 0.0148)'$.

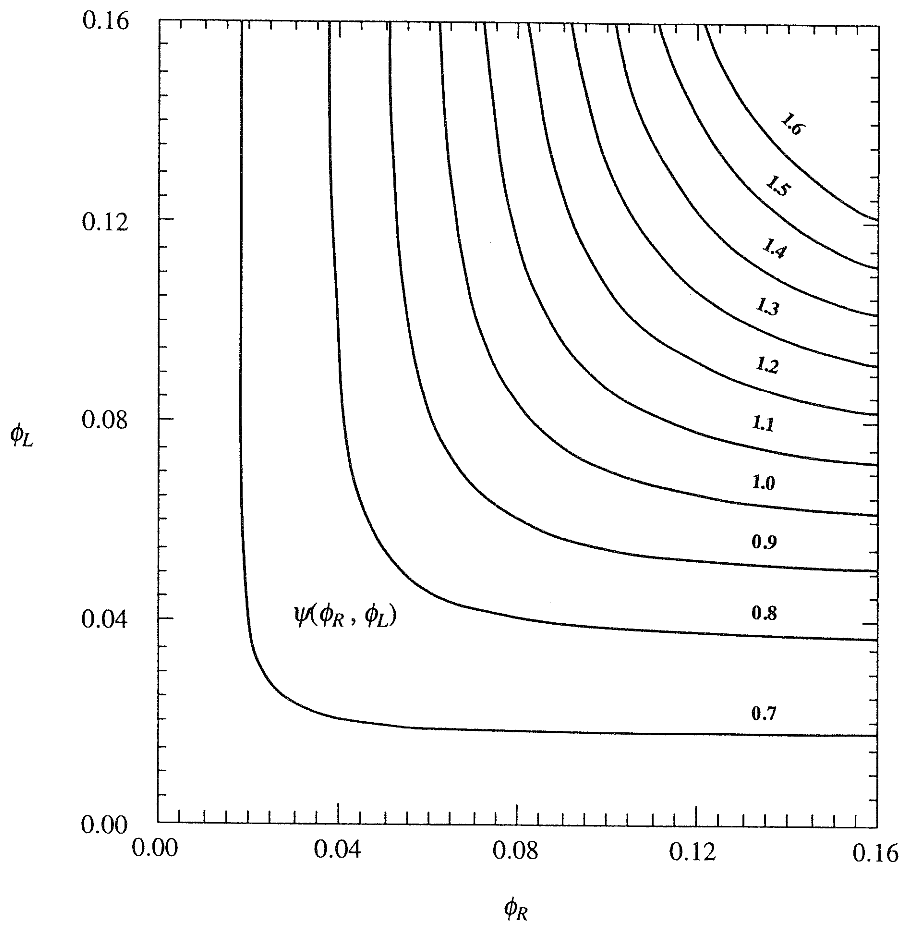

Figure 4. The contour plot of $\psi = \psi(\phi_R, \phi_L) = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \phi)$. There are many combinations of ϕ_L and ϕ_R which produce the same minimum output variance, $\psi(\phi_R, \phi_L)$. Thus, the tolerance design problem is to choose the most economical combination of ϕ_L and ϕ_R. However, once these are chosen, the optimal parameter values must be recalculated to achieve the minimum output variance.

Because the optimal parameter values change when the tolerances change, it follows that the designer should either consider the parameter and tolerance design together or follow an iterative process that will adjust the parameter values whenever tolerance values are changed. If this is not done, the design will be suboptimal.

In addition to raising this important philosophical point, Figure 4 can be used to show the trade-offs that must be made between the tolerances of the design variables and the output variance. Note that in our example, the relationship between ψ and ϕ_L is similar to the relationship between ψ and ϕ_R, and hence symmetric about $\phi_L = \phi_R$. The designer may therefore maintain the level of variance in the current by relaxing the tolerance on one of the components and correspondingly tightening the tolerance of the other. In addition, the plot shows the designer how much the tolerances must be tightened to achieve a lower variance in current.

To produce the plot in Figure 4 is computer-intensive. Each coordinate point involves the solution of a constrained optimization problem. It is, however, of important conceptual value. Any optimization ought to be accompanied by various sensitivity analyses. One kind of sensitivity analysis involves calculations of the change in the objective function as a function of changing the input variables $\mathbf{x}$ from the optimal value $\mathbf{x}^o$. Another equally important but less common type of sensitivity analysis involves a calculation of how the optimal solution $\mathbf{x}^o$ changes as the assumptions change. We believe that the rapid developments in computing power will soon make this second kind of sensitivity analysis realistic for more complex parameter design problems.

Conclusion

We have provided a general formulation of the problem of parameter design and have shown how standard methods of nonlinear programming theory can be applied to provide additional insight into the problem's solution. We have also shown that the optimal solution depends on the specified standard deviations and, hence, on the assumed tolerances. It is, therefore, not possible to solve the parameter design and tolerance design problems separately as often assumed in the past; these two problems must be solved simultaneously.

The simple example using a *RL* circuit is clearly not very interesting in itself as a design problem. However, as a vehicle for discussion of the concepts and the geometry of the parameter design problem, this simple illustration has been most useful. In particular, we found it conceptually mind-expanding to consider the function of conditional minima $\psi(\boldsymbol{\sigma}_\mathbf{x}) = \min_\mathbf{x} L(\mathbf{x}, \lambda, \boldsymbol{\sigma}_\mathbf{x})$ as a function of the input standard deviations, $\boldsymbol{\sigma}_\mathbf{x} = (\sigma_1, \sigma_2, \ldots, \sigma_n)'$, shown in Figure 4.

References

1. Taguchi, G. and Wu, Y., *Introduction to Off-Line Quality Control*, Central Japan Quality Control Association, Nagoya, 1985.
2. Kacker, R. N., Off-Line Quality Control, Parameter Design, and the Taguchi Method, *J. Qual. Technol.*, 17, 176–209 (1985).
3. Box, G. E. P. and Fung, C. A., Studies in Quality Improvement: Minimizing Transmitted Variation by Parameter Design," Technical Report 8, Center for Quality and Productivity Improvement, Madison, WI (1986).
4. Vining, G. G. and Myers, R. H., Combining Taguchi and Response Surface Philosophies, *J. Qual. Technol.*, 22, 38–45 (1990).
5. Myers, R. H., Khuri, A. I., and Vining, G., Response Surface Alternatives to the Taguchi Robust Parameter Design Approach, *The American Statistician*, 46, 131–139 (1992).

6. Del Castillo, E. D. and Montgomery, D. C., A Nonlinear Programming Solution to the Dual Response Problem, *J. Qual. Technol.*, 25, 199–204 (1993).
7. Taguchi, G., *An Introduction to Quality Engineering*, UNIPUB/Kraus International, White Plains, NY, 1986.
8. Taguchi, G., Development of Quality Engineering, *ASI J.*, 1, 5–29 (1988).
9. Nair, V. N. (ed.), Taguchi's Parameter Design: A Panel Discussion, *Technometrics*, 34, 127–161 (1992).
10. Morrison, S. J., The Study of Variability in Engineering Design, *Appl. Statist.*, 6, 133–138 (1957).
11. Wilde, D. J., Monotonicity Analysis of Taguchi's Robust Circuit Design Problem, in *Proceedings, ASME 1990 Advances in Design Automation*, ASME, New York, 1990, pp. 75–80.
12. Hsieh, C. C., Oh, K. P., and Oh, H. L., A Design Technique for Minimizing Net Build Errors, in *Proceedings, ASME 1990 Winter Annual Meeting*, ASME, New York, 1990, pp. 61–69.
13. Arora, J. S., *Introduction to Optimum Design*, McGraw-Hill, New York, 1989.
14. Parkinson, A., Sorensen, C., Free, J., and Canfield, B., Tolerances and Robustness in Engineering Design Optimization, in *Proceedings, ASME 1990 Advances in Design Automation*, ASME, New York, 1990, pp. 121–128.
15. Sundaresan, S., Ishii, K., and Houser, D. R., Design Optimization for Robustness using Performance Simulation Programs, in *Proceedings, ASME 1991 Advances in Design Automation*, ASME, New York, 1991, pp. 249–256.
16. Michelena, N. F. and Agogino, A. M., Formal Solution of *N*-Type Taguchi Parameter Design Problems with Stochastic Noise Factors, in *Proceedings of ASME 1991 Design Theory and Methodology*, ASME, New York, 1991, pp. 13–20.
17. Ramakrishnan, B. and Rao, S. S., A Robust Optimization Approach Using Taguchi's Loss Function for Solving Nonlinear Optimization Problems, in *Proceedings, ASME 1991 Advances in Design Automation*, ASME, New York, 1991, pp. 241–248.
18. Luenberger, D. G., *Linear and Nonlinear Programming*, 2nd ed., Addison-Wesley, Reading, MA, 1984.
19. Ku, H. H., Notes on the Use of Propagation of Error Formulas, in *Precision Measurement and Calibration, Selected NBS Papers on Statistical Concepts and Procedures*, edited by H. H. Ku, Government Printing Office, Washington, DC, 1966.
20. Fathi, Y., A Nonlinear Programming Approach to the Parameter Design Problem, *Eur. J. Operat. Res.*, 53, 371–381 (1991).
21. Courant, R., *Differential and Integral Calculus*, Vol. II, Interscience, New York, 1959.
22. Box, G. E. P., and Fung, C. A., Quality Quandaries: Is Your Robust Design Procedure Robust? *Qual. Eng.*, 7 (1994).

About the Authors: Søren Bisgaard is the director at the Center for Quality and Productivity Improvement and associate professor in the Department of Industrial Engineering, University of Wisconsin–Madison. He holds two engineering degrees in industrial and manufacturing engineering and M.S. and Ph.D. degrees in statistics from the University of Wisconsin–Madison. Before entering the university, he worked for several years as a machinist/tool maker for companies in Denmark and Germany. Dr. Bisgaard has worked as an industrial consultant in quality improvement, operations research, and managerial accounting. He is a co-author with George Box and Conrad Fung of the video

series "Designing Industrial Experiments." Dr. Bisgaard's research interests are in the areas of experimental design, the use of statistics for designing better quality into products and processes, quality control, quality management, and concurrent engineering. He has received both the Shewell and Brumbaugh Awards from the American Society for Quality Control. In 1990, Dr. Bisgaard received the Ellis R. Ott Award for excellence in quality improvement.

Bruce Ankenman is a research assistant at the Center for Quality and Productivity Improvement and a Ph.D. student in the Quality Engineering Program in the Department of Industrial Engineering, University of Wisconsin–Madison. He holds a B.S. degree in electrical engineering from Case Western Reserve University and an M.S. degree in Manufacturing Systems Engineering from University of Wisconsin–Madison. Before returning for graduate studies, he worked for 5 years as a product design engineer for an automotive supplier in Ohio.