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NON-PARAMETRIC TOLERANCE INTERVALS FOR SMALL SAMPLE SIZES

An empirical likelihood approach

Agenda

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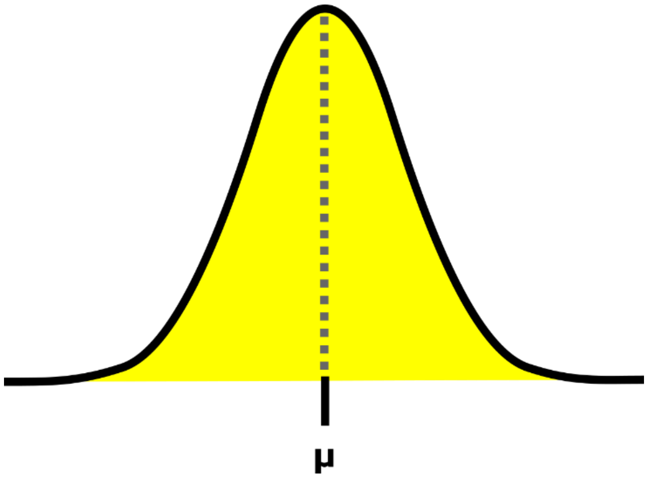
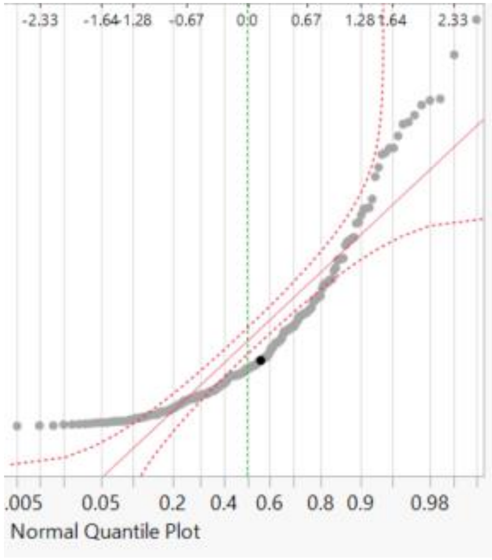
Tolerance Intervals

- Why Tolerance Intervals?
  - Upper limit for failures
  - Statement about (almost) all future observations

- Why non-parametric?
  - Normal distribution is common but not universal

- Why small samples?
  - Generating data can be costly
  - Destructive tests are necessary





Problems

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Demonstration in JMP

Problems: Summary

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- Tolerance Intervals
  - Unable to calculate non-parametric tolerance Intervals for small sample sizes
- Idea: Calculate confidence intervals for quantiles.
  - Non-parametric approach from JMP: empirical likelihood
  - Problem: Confidence are not usable for extreme Quantiles and small sample sizes

Approach

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- How does smoothed empirical likelihood Work?
  - Are there different methods?
- How does JMP calculate the confidence Intervals?
- Is there a way to eliminate the problems with small sample sizes and extreme Quantiles?

Existing Methods

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- Empirical likelihood was first introduced by Owen (1988)

- When quantiles are considered, the log likelihood is dependent on the empirical distribution function F_n (Adimari, 1998):

$$l(\Theta) = 2n\left[F_n(\Theta)log\left(\frac{F_n(\Theta)}{q}\right) + (1 - F_n(\Theta))log\left(\frac{1 - F_n(\Theta)}{1 - q}\right)\right]$$

- Confidence intervals can be calculated by using Wilks theorem Owen (1988): $\lim_{n \to \infty} P(l(\Theta) \leq c) = P(\chi_1^2 \leq c)$

- Several methods of smoothing exist:
  - Smoothing using a kernel function. (Chen, Hall, 1993) (JMP)
  - Linear smoothing of F_n (Adimari, 1998)

Comparison. Median

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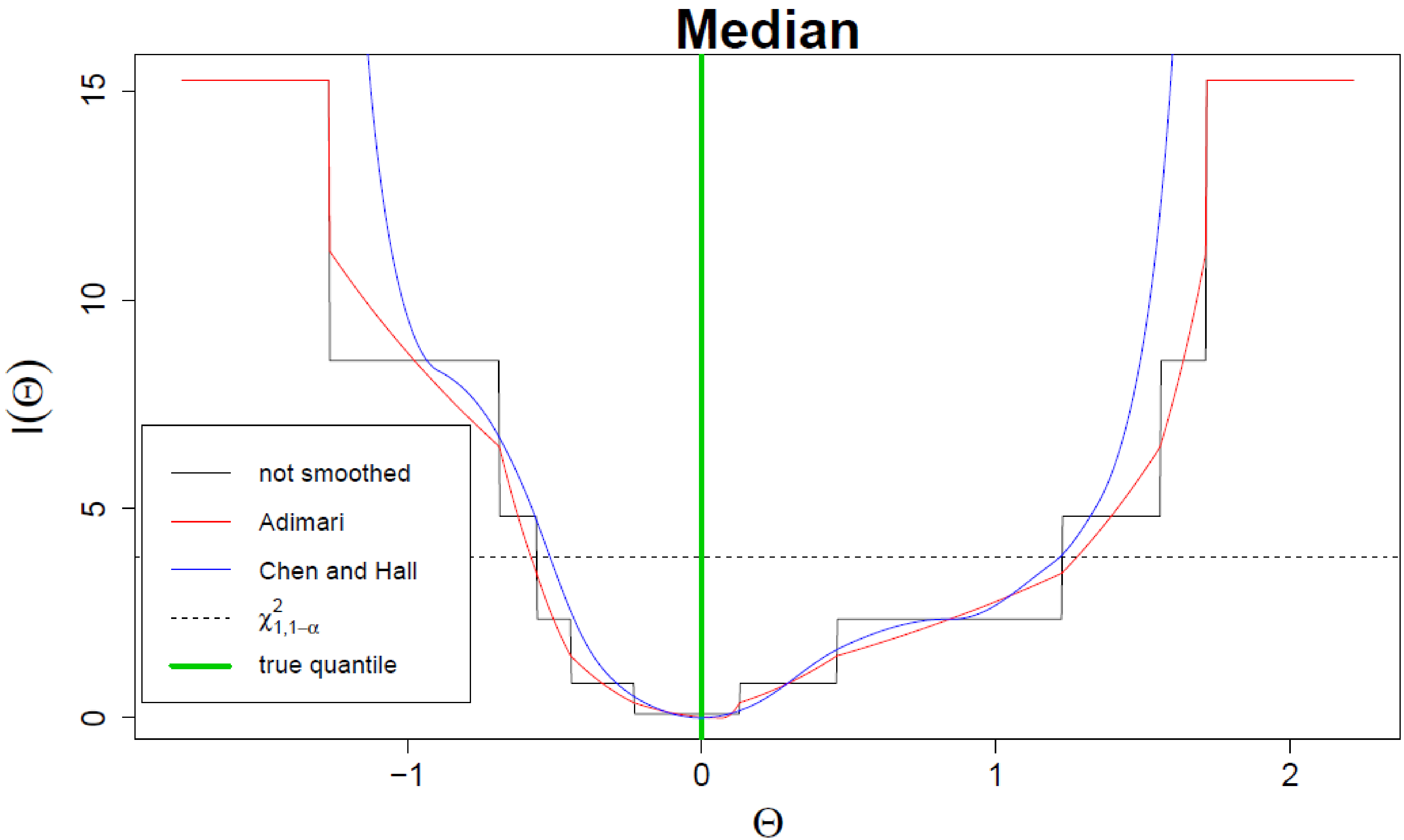

Figure: Smoothed empirical likelihood functions $(n = 11; q = 0.5; \alpha = 0.05)$

Comparison 1%-Quantile

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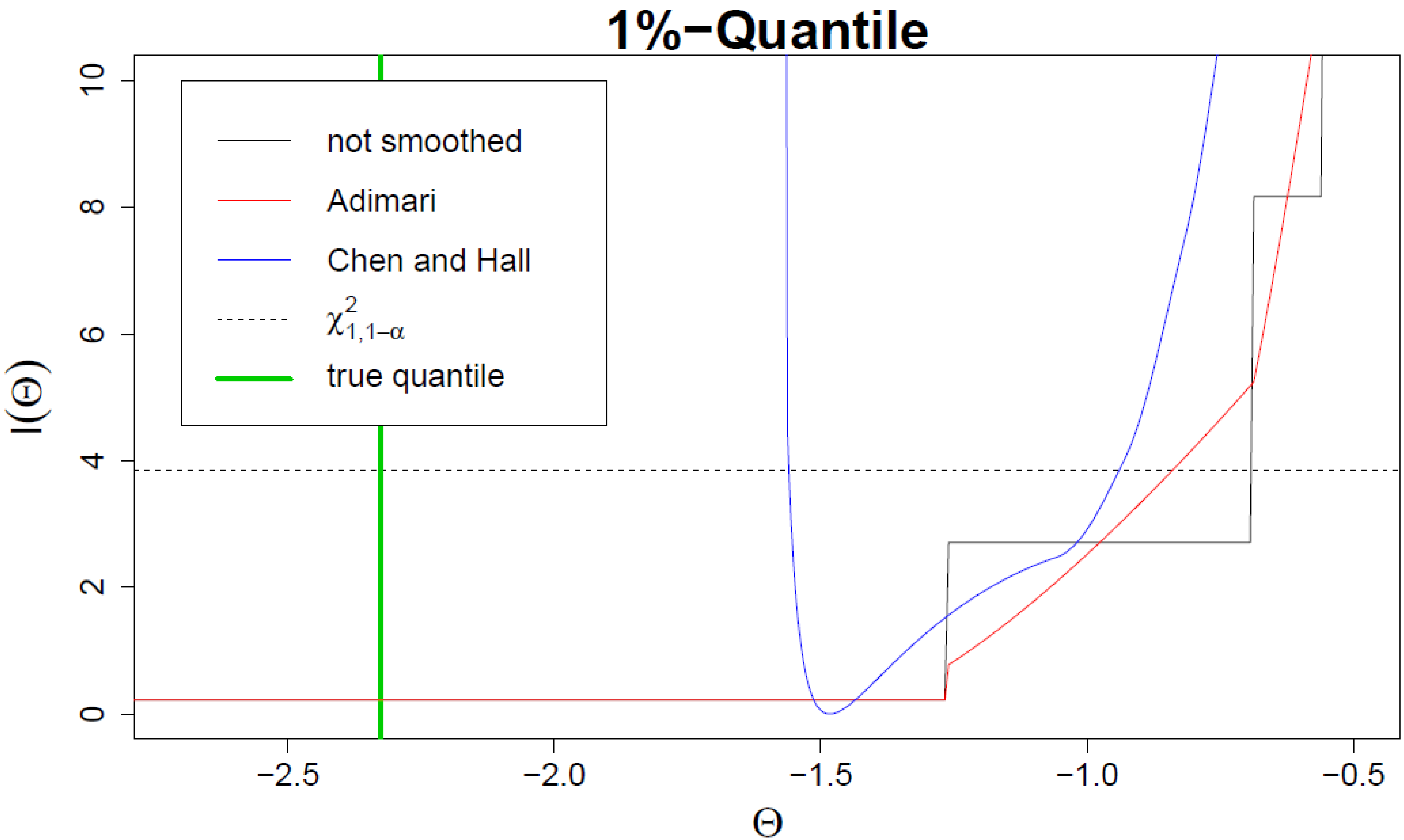

Figure: Smoothed empirical likelihood functions ($n = 11; q = 0.01; \alpha = 0.05$)

Coverage Rates

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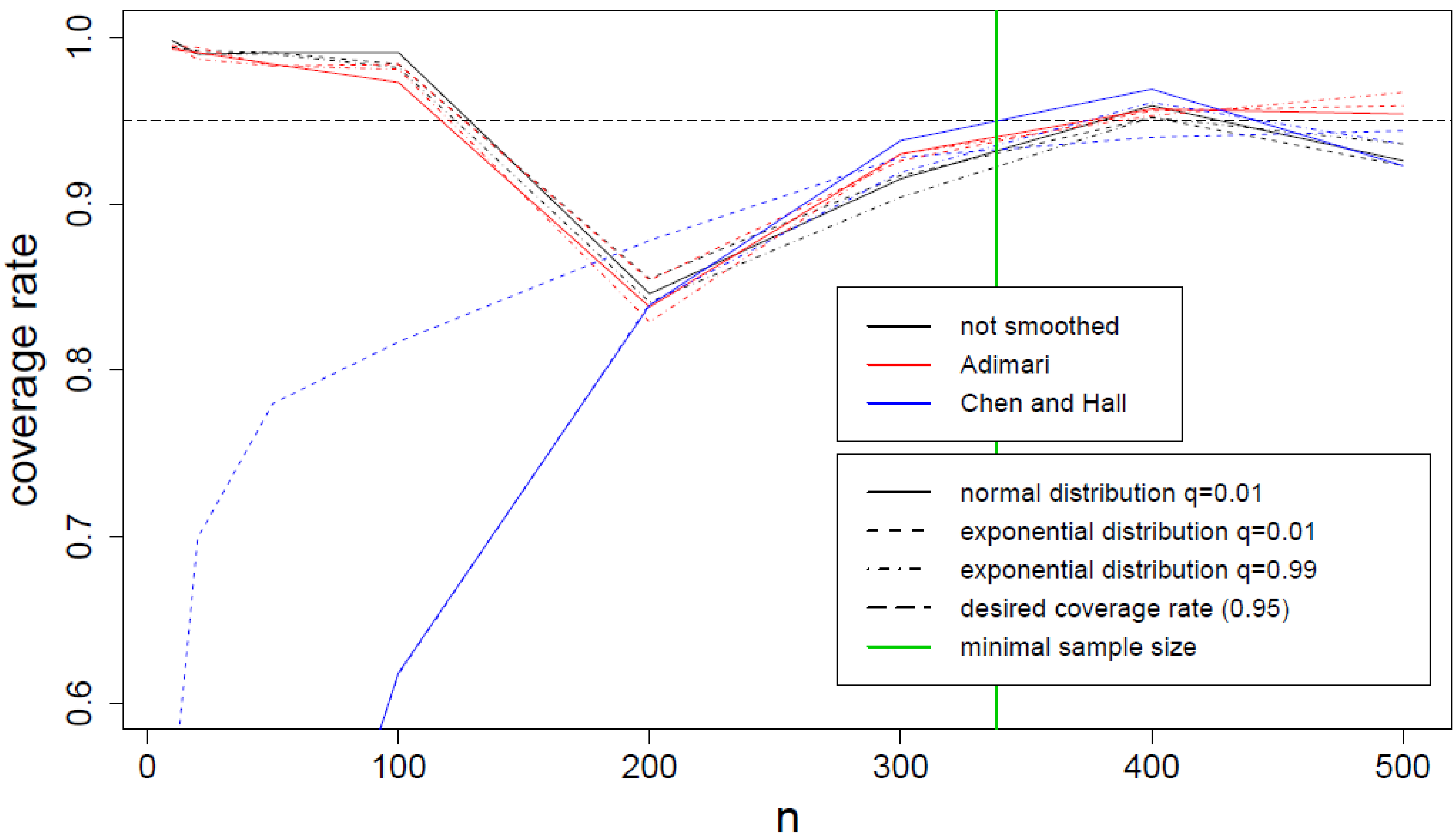

Figure: Estimated coverage rates based on 1000 samples

Linear smoothing

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- The smoothing proposed by Adimari (1998) is achieved by using a linear smoothing F^* of F_n:

$$F_n^*(\Theta) = \begin{cases} 0 & \text{if } \Theta < x_{(1)} \\ H(\Theta) & \text{if } \Theta \in [x_{(1)}, x_{(n)}) \\ 1 & \text{if } \Theta \geq x_{(n)} \end{cases}$$

where

$$H(\Theta) = \begin{cases} \frac{2i-1}{2n} & \text{if } \Theta = x_{(i)}; i \in \{1,..,n-1\} \\ (1-\lambda)\frac{2i-1}{2n} + \lambda\frac{2i+1}{2n} & \text{if } \Theta \in (x_{(i)}, x_{(i+1)}); \lambda = \frac{\Theta - x_{(i)}}{x_{(i+1)} - x_{(i)}}; i \in \{1,..,n-1\} \end{cases}$$

Empirical distribution function

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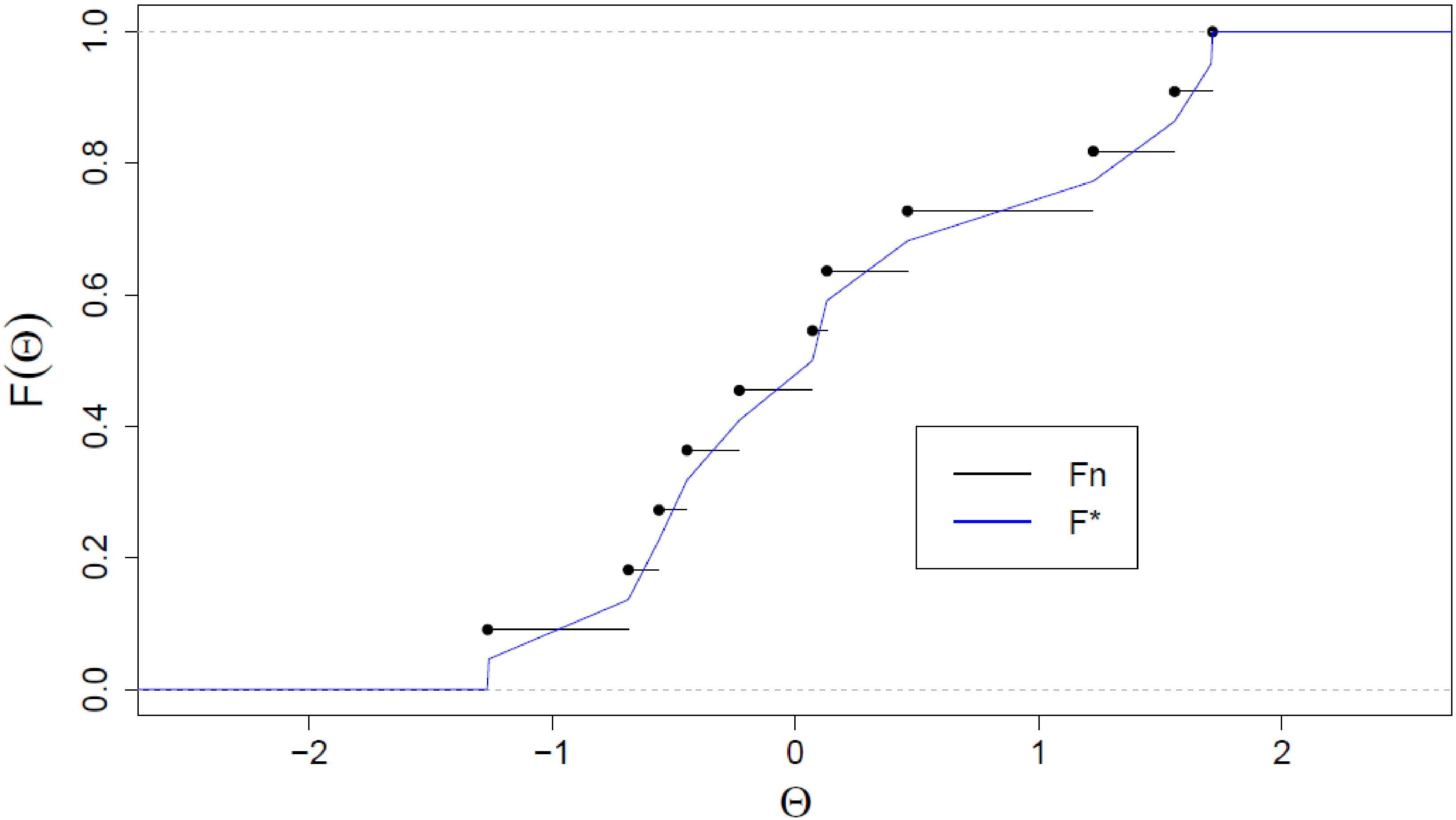

Figure: Smoothing of F_n for 11 observations from a standard normal distribution

Extending the empirical distribution function

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- Find an extension of the likelihood function for values outside of the observed data so that
  - finite confidence intervals are guaranteed.
  - the desired coverage rate is achieved.

1. extending F^* as follows:

$$F_{ext}(\Theta) = \begin{cases} 0 & \text{if } \Theta \leq x_{(1)} - d_1 c \\ \frac{1}{2n} - \frac{1}{2n * d_1 * c}(x_{(1)} - \Theta) & \text{if } x_{(1)} - d_1 c < \Theta < x_{(1)} \\ H(\Theta) & \text{if } x_{(1)} \leq \Theta \leq x_{(n)} \\ \frac{2n-1}{2n} + \frac{1}{2n * d_2 * c}(\Theta - x_{(n)}) & \text{if } x_{(n)} < \Theta < x_{(n)} + d_2 c \\ 1 & \text{if } \Theta \geq x_{(n)} + d_2 c \end{cases}$$

Where $c \geq 1$; $d_1 = \frac{1}{10} \sum_{i=1}^{5} (x_{(i+1)} - x_{(i)})$ and
$d_2 = \frac{1}{10} \sum_{i=1}^{5} (x_{(n-i+1)} - x_{(n-i)})$

2. linear extension of the likelihood function.

Example

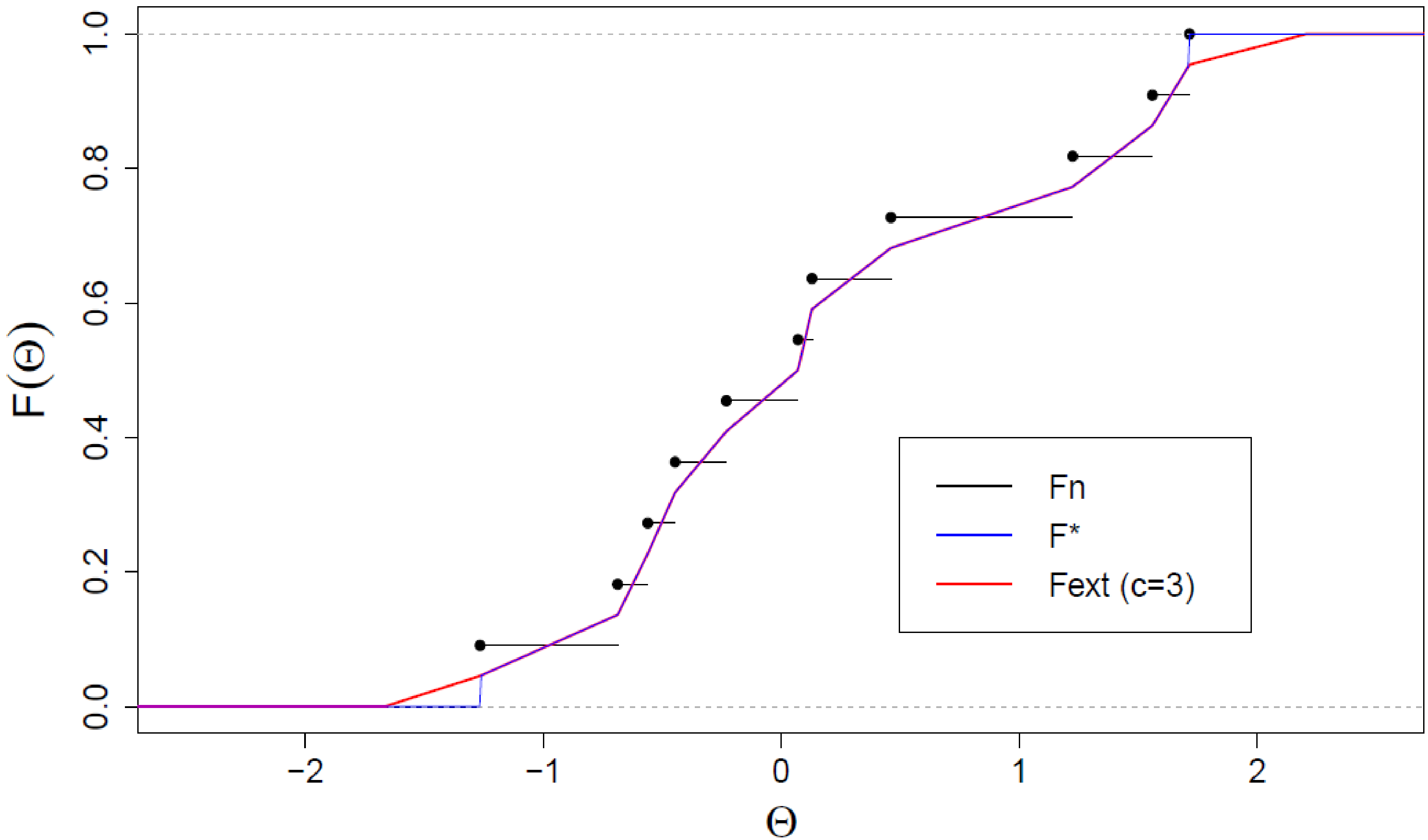

Figure: Smoothing of F_n using F_{ext}

Example

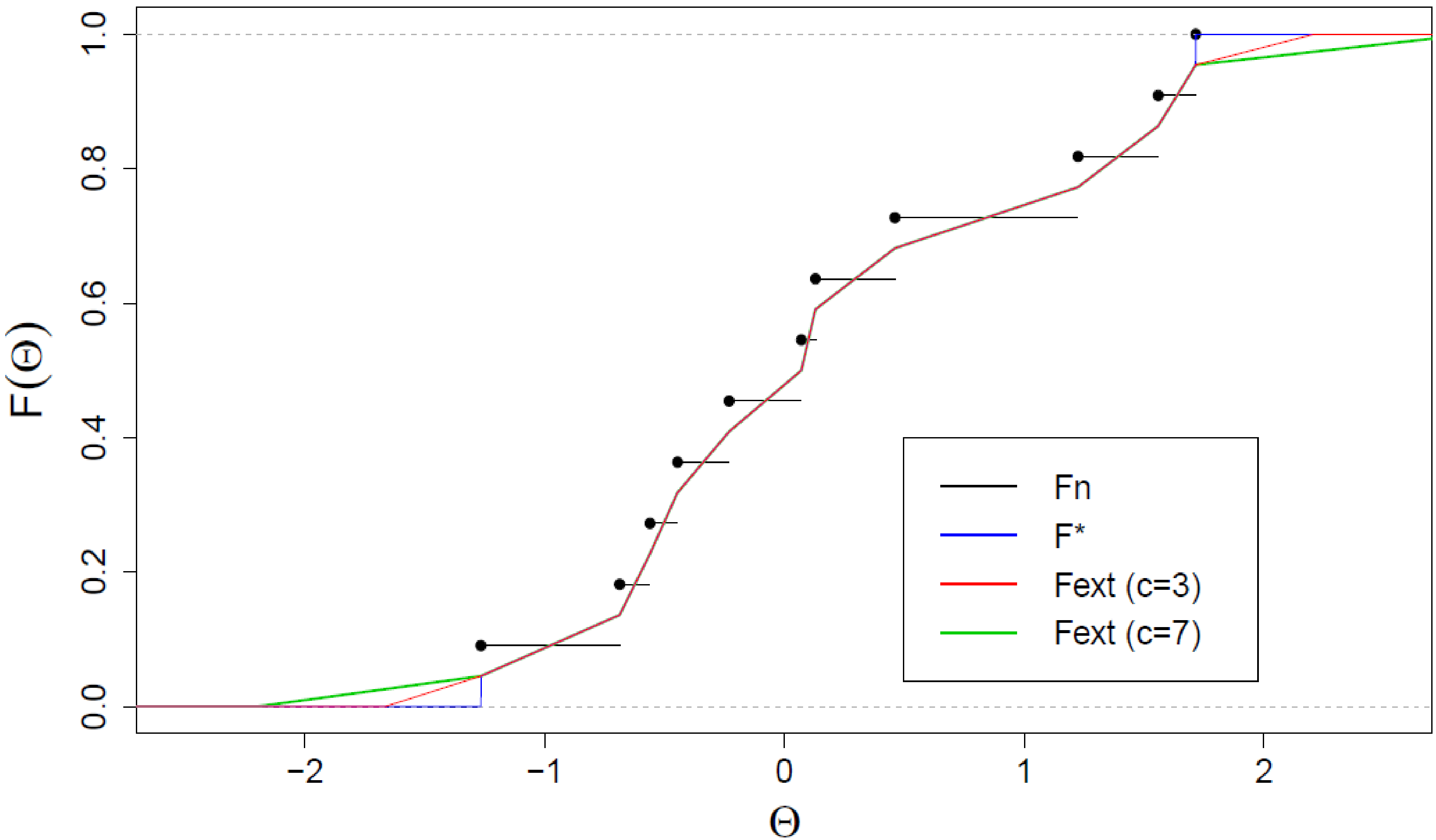

Figure: Visualising the influence of the parameter c on F_{ext}

Example

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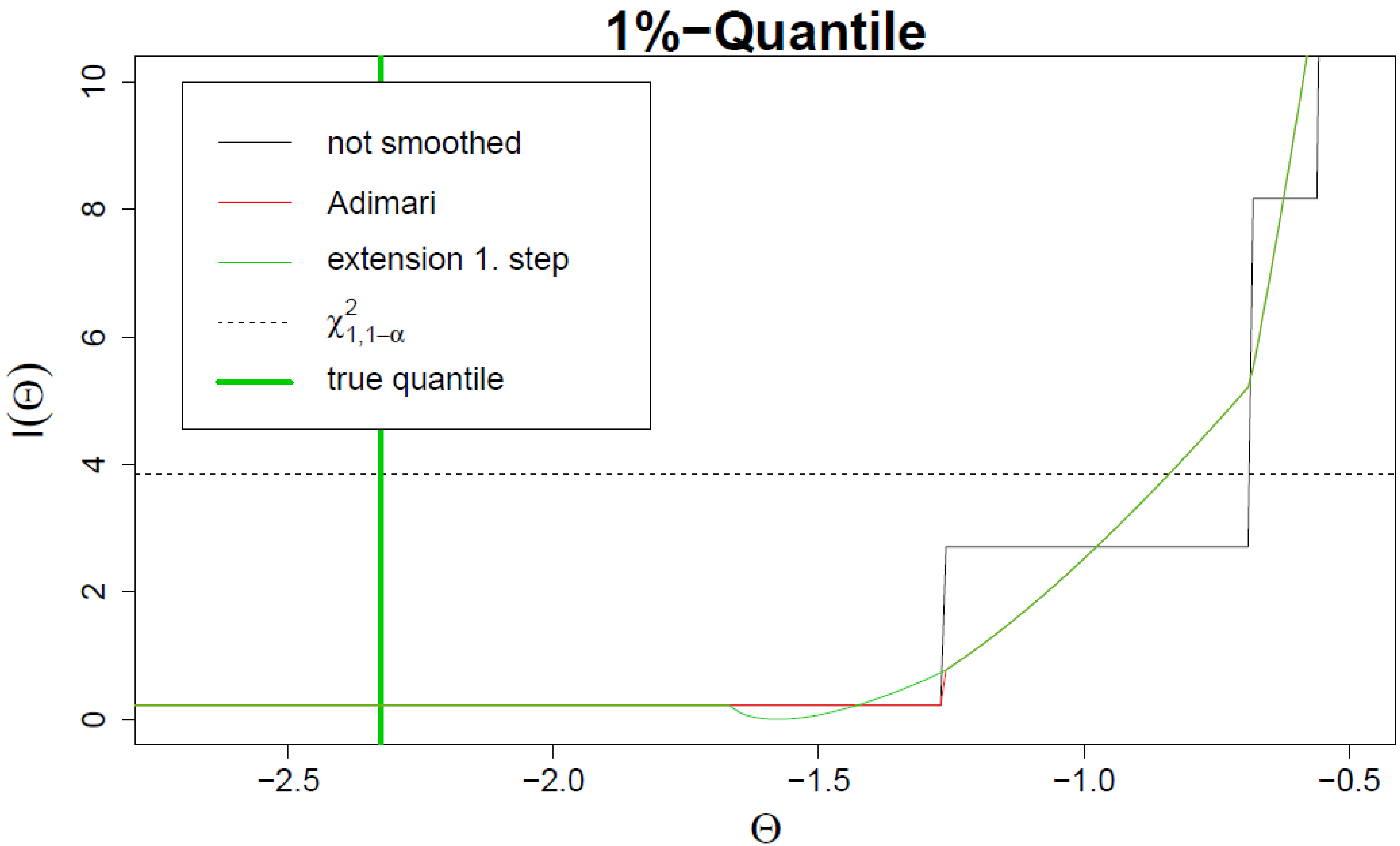

Figure: First step of the extension $(c = 3)$

Example

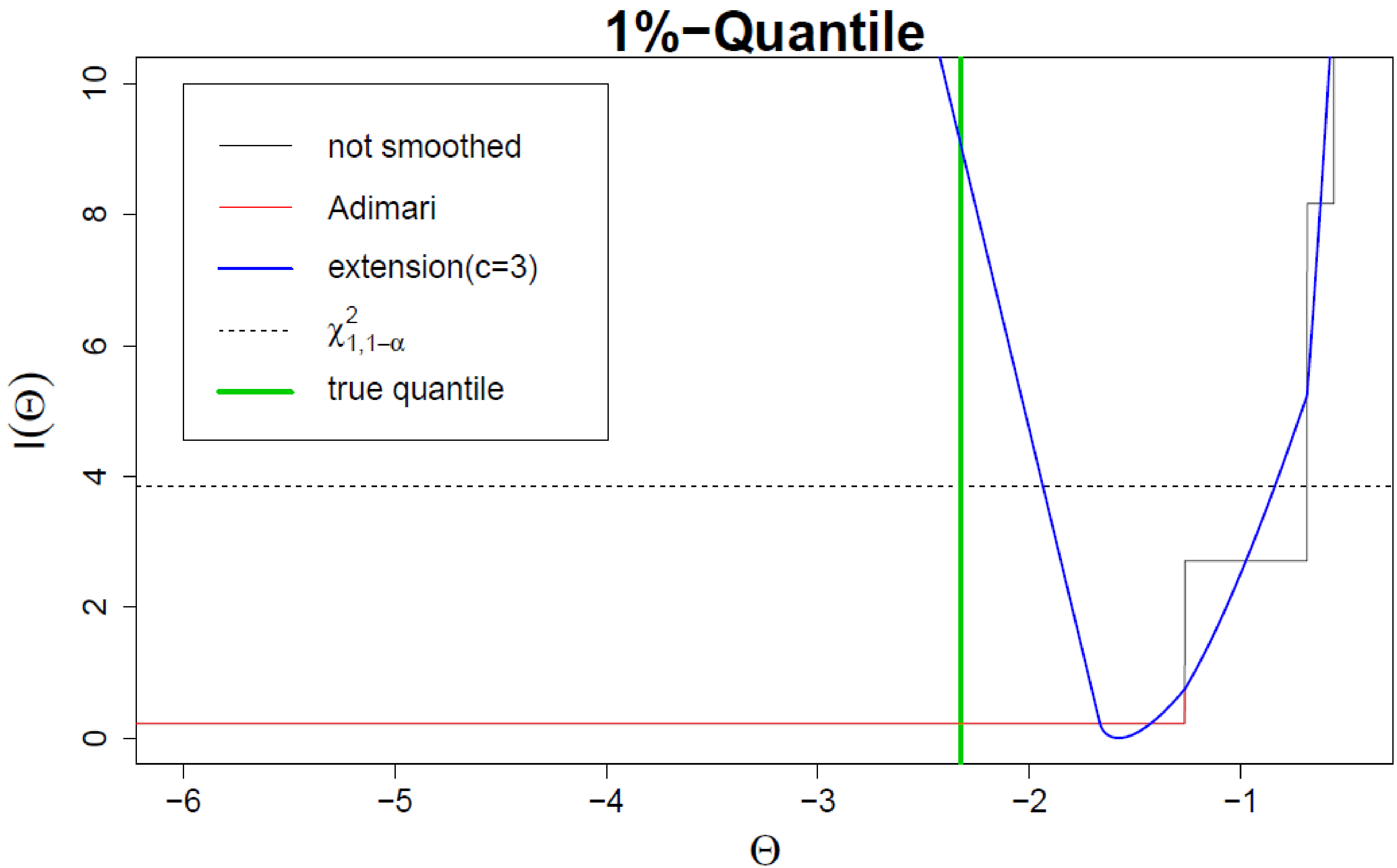

Figure: Second step: further linear extension $(c = 3)$

Example

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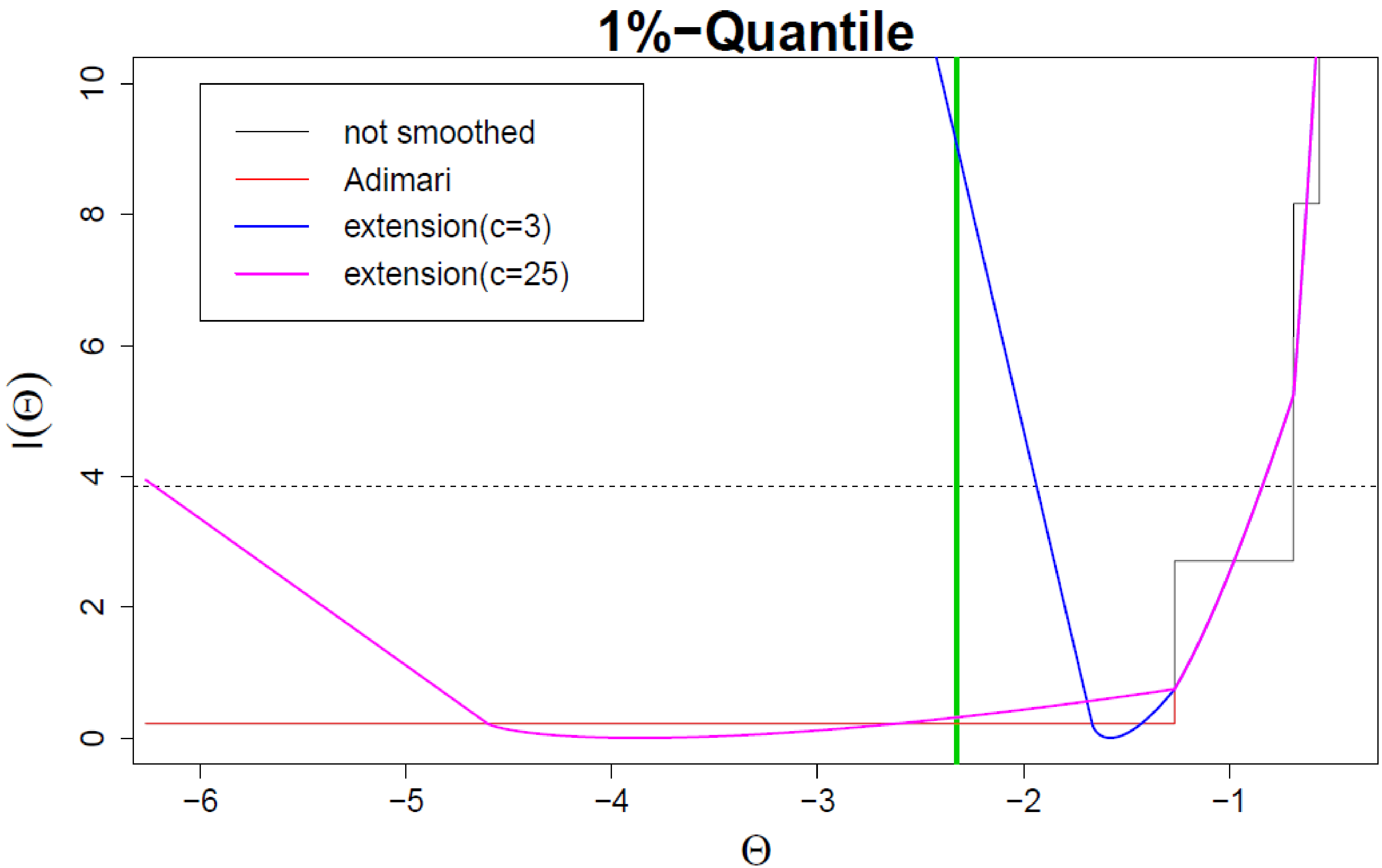

Figure: Fully extended likelihood function for different values of c

Assuring Coverage

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- The quality of the Confidence Interval is dependent on the extension parameter c.
- The smallest value of c which results in a coverage rate of at least $(1-\alpha)$ is desired.

- A simulation study was carried out under the following assumpitons:
  - The required value for c depends on q, n, R and α

$$R := \begin{cases} q * n & \text{if } q \leq 0.5 \\ (1-q) * n & \text{if } q > 0.5 \end{cases}$$

  - The required value for c does not depend on the distribution of the data.

Simulation Study

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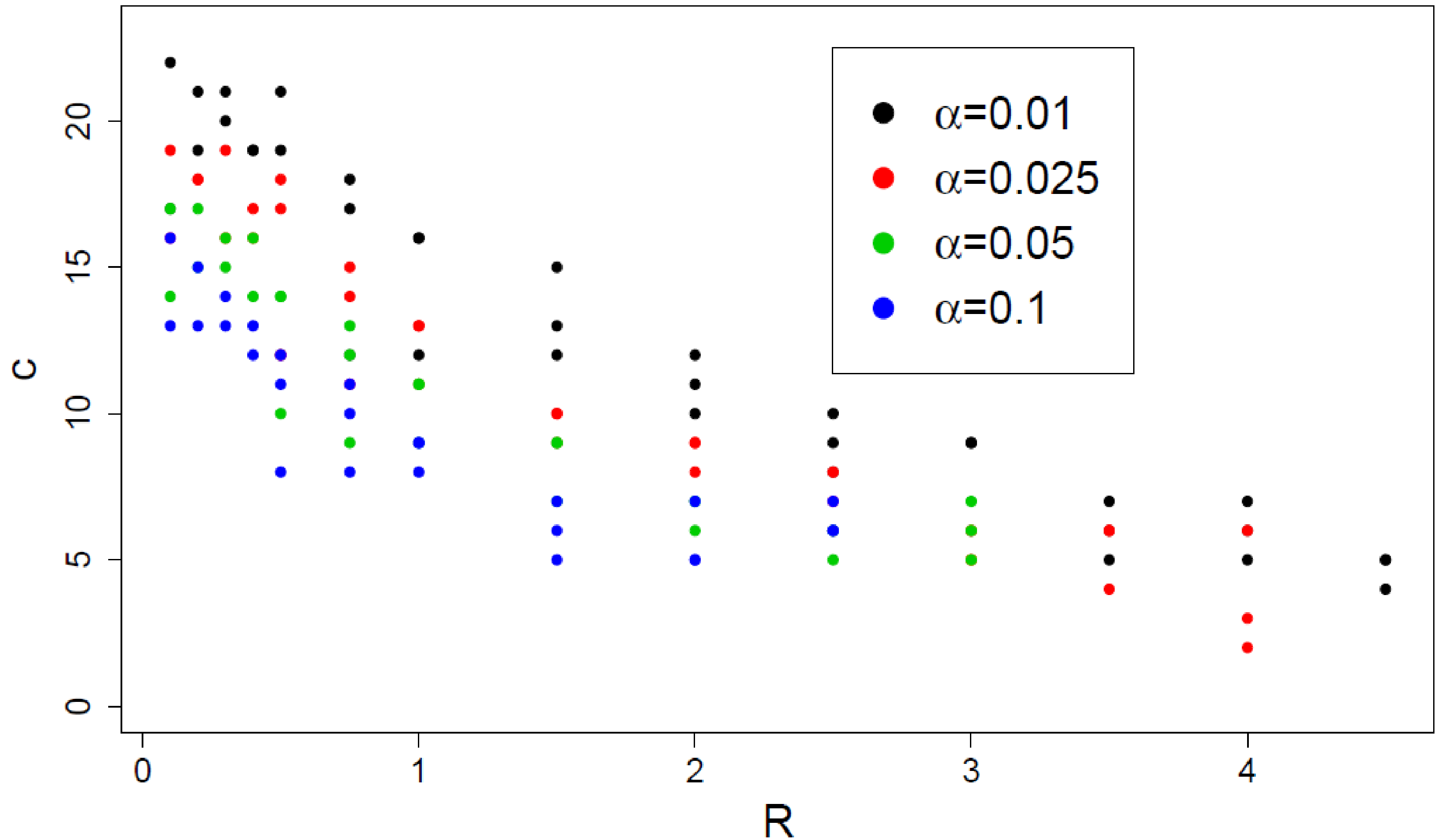

Figure: Chosen value of c for different values of R

Simulation Study

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- Model for c: $\hat{c} = 12.344 - 7.082\sqrt{R} - 2.454\log(\alpha) - 75.125q - 0.004n$
- Adjusted $R^2 = 0.933$

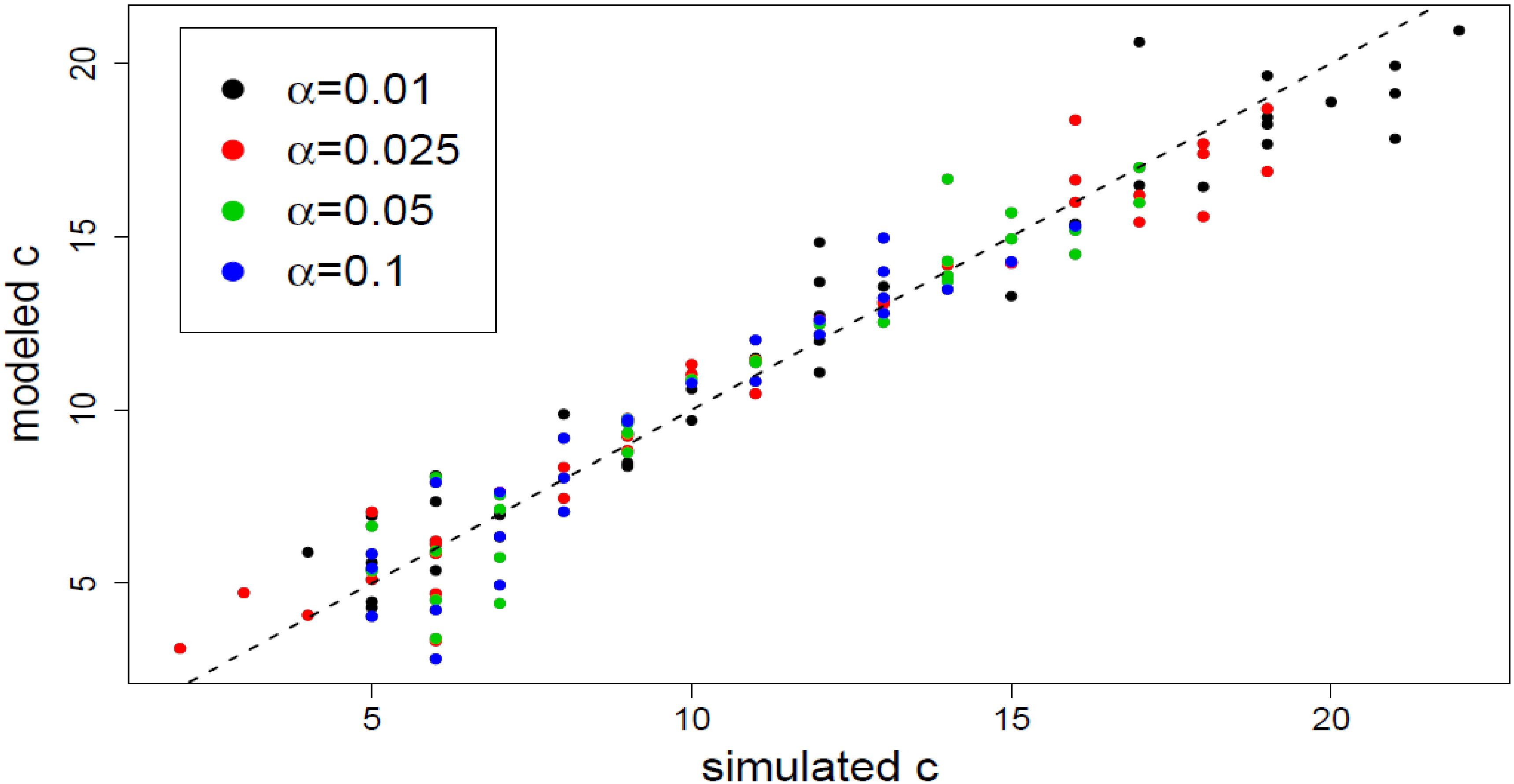

Figure: Visualising the Model

Example

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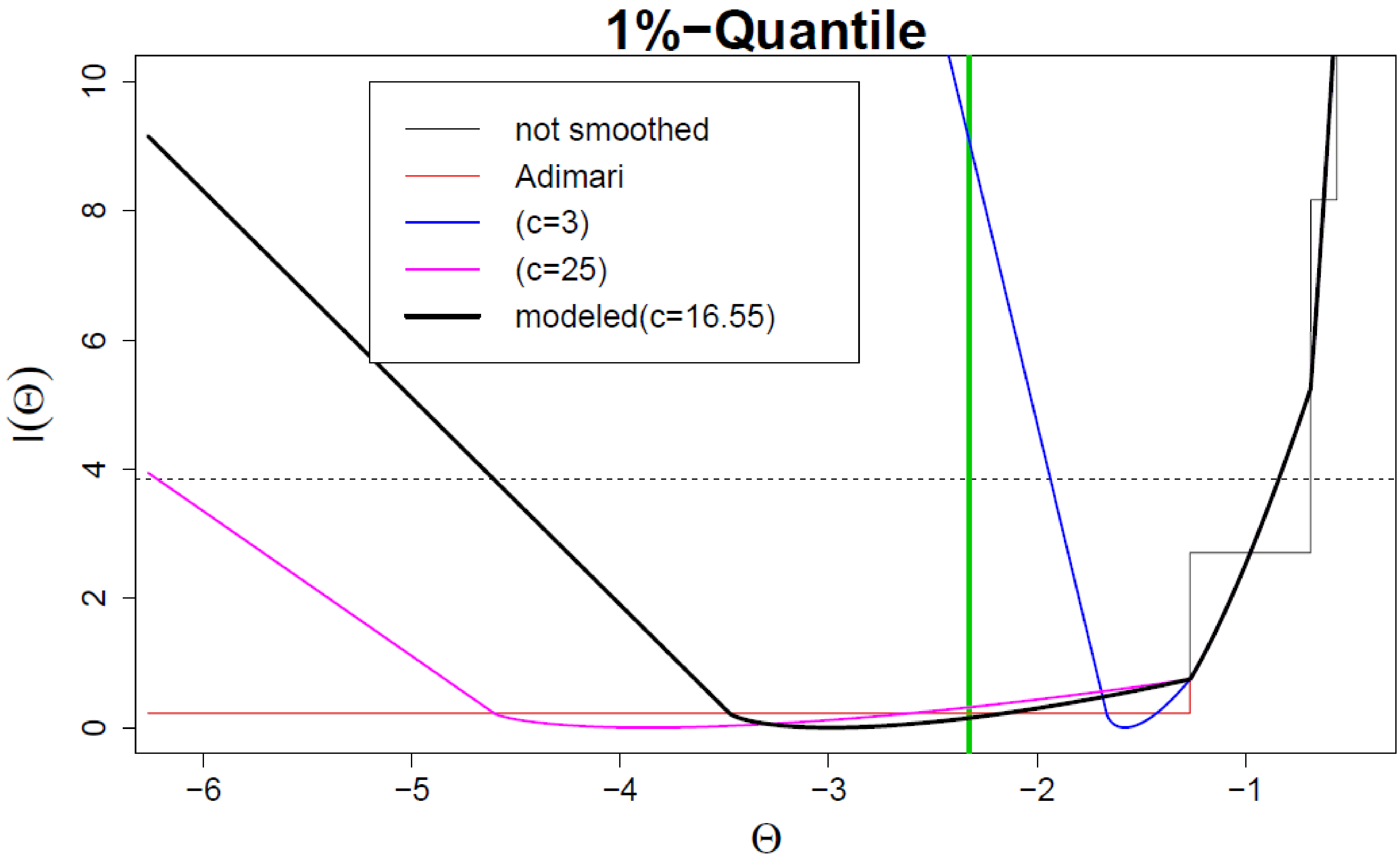

Figure: Example for the modeled value of c ($n = 11$, $q = 0.01, \alpha = 0.05$)

Coverage Rates

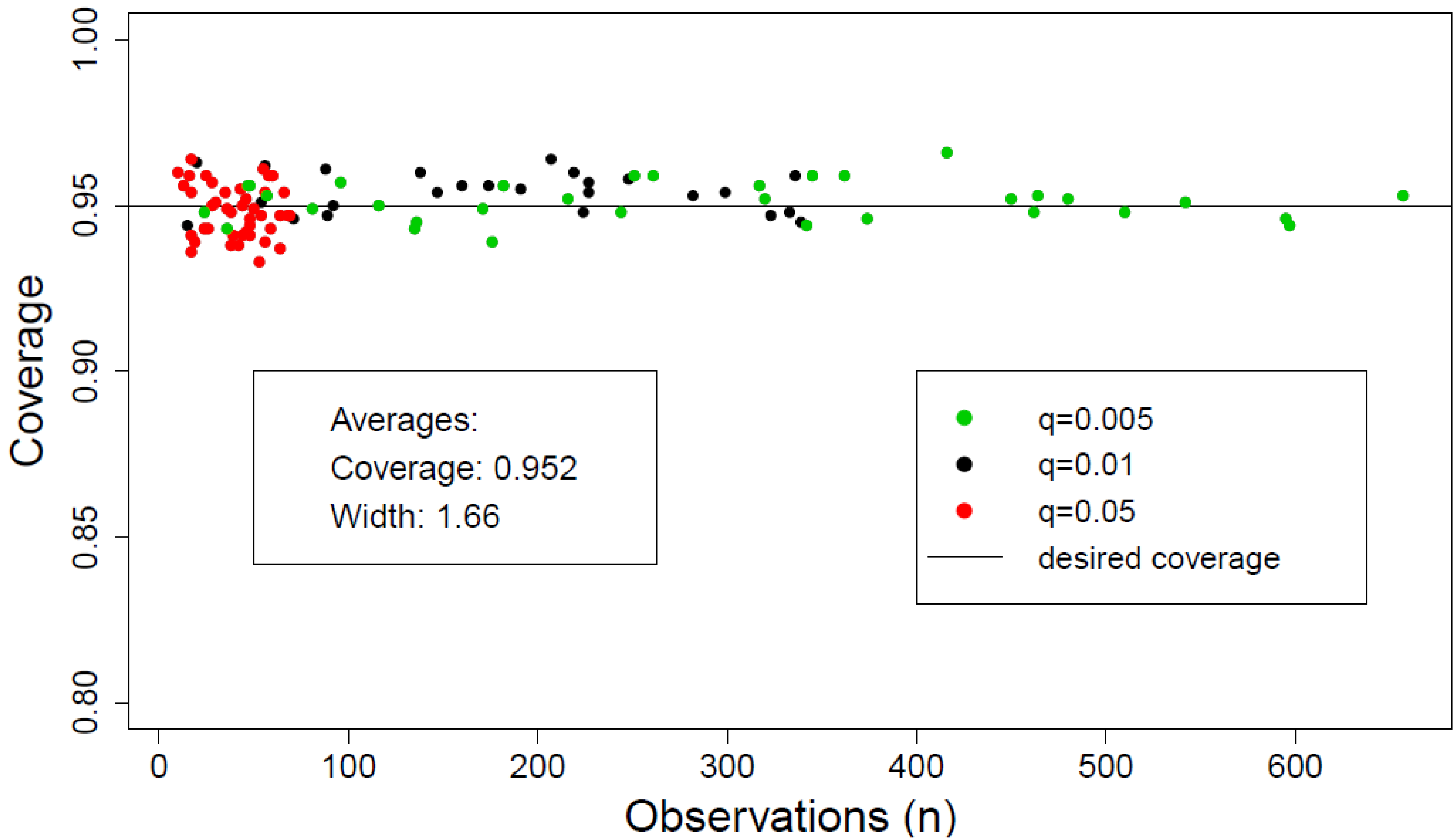

Figure: coverage rates based on 1000 samples for normally distributed data

Coverage Rates

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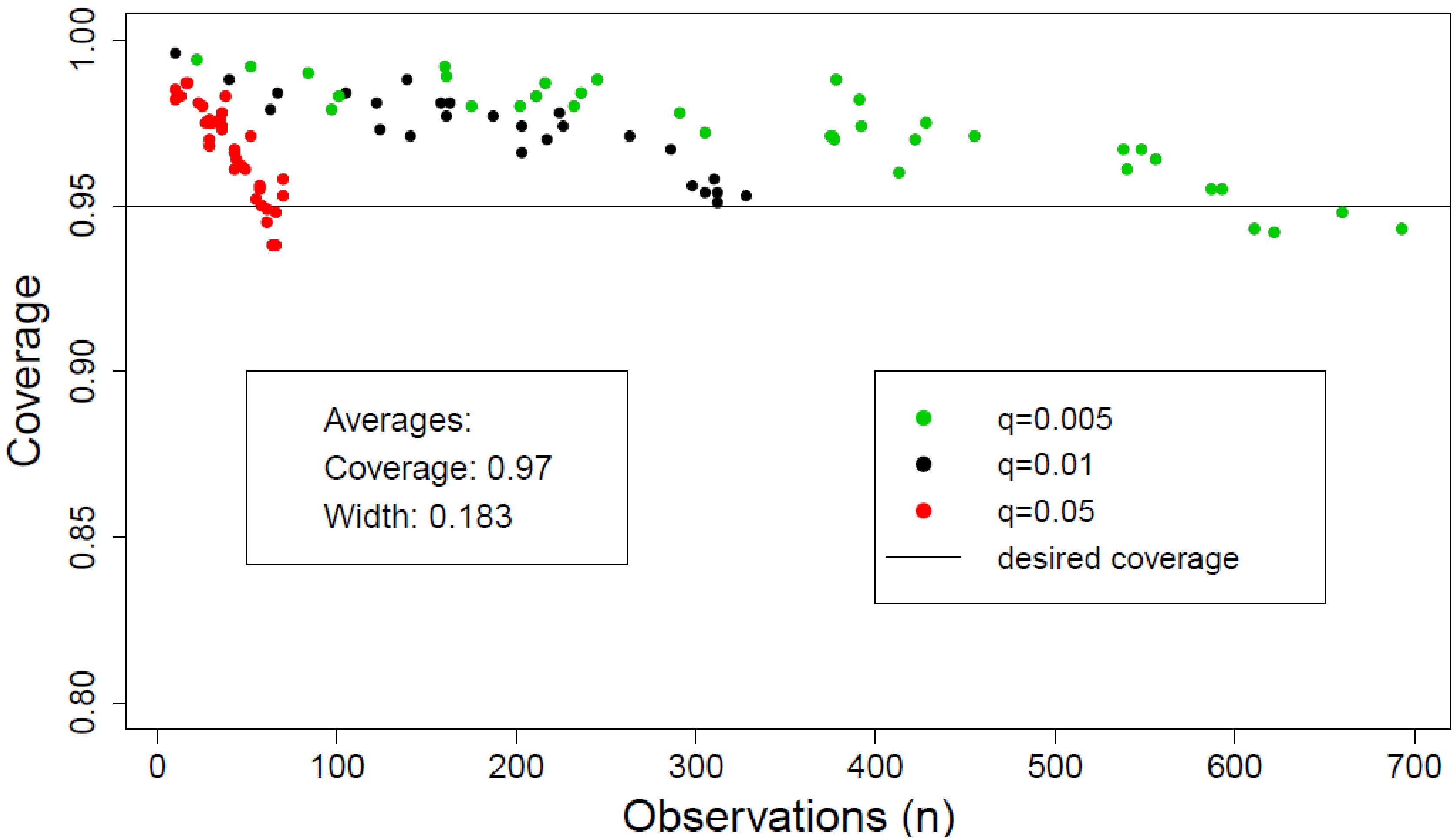

Figure: coverage rates based on 1000 samples for exponentially distributed data

Coverage Rates

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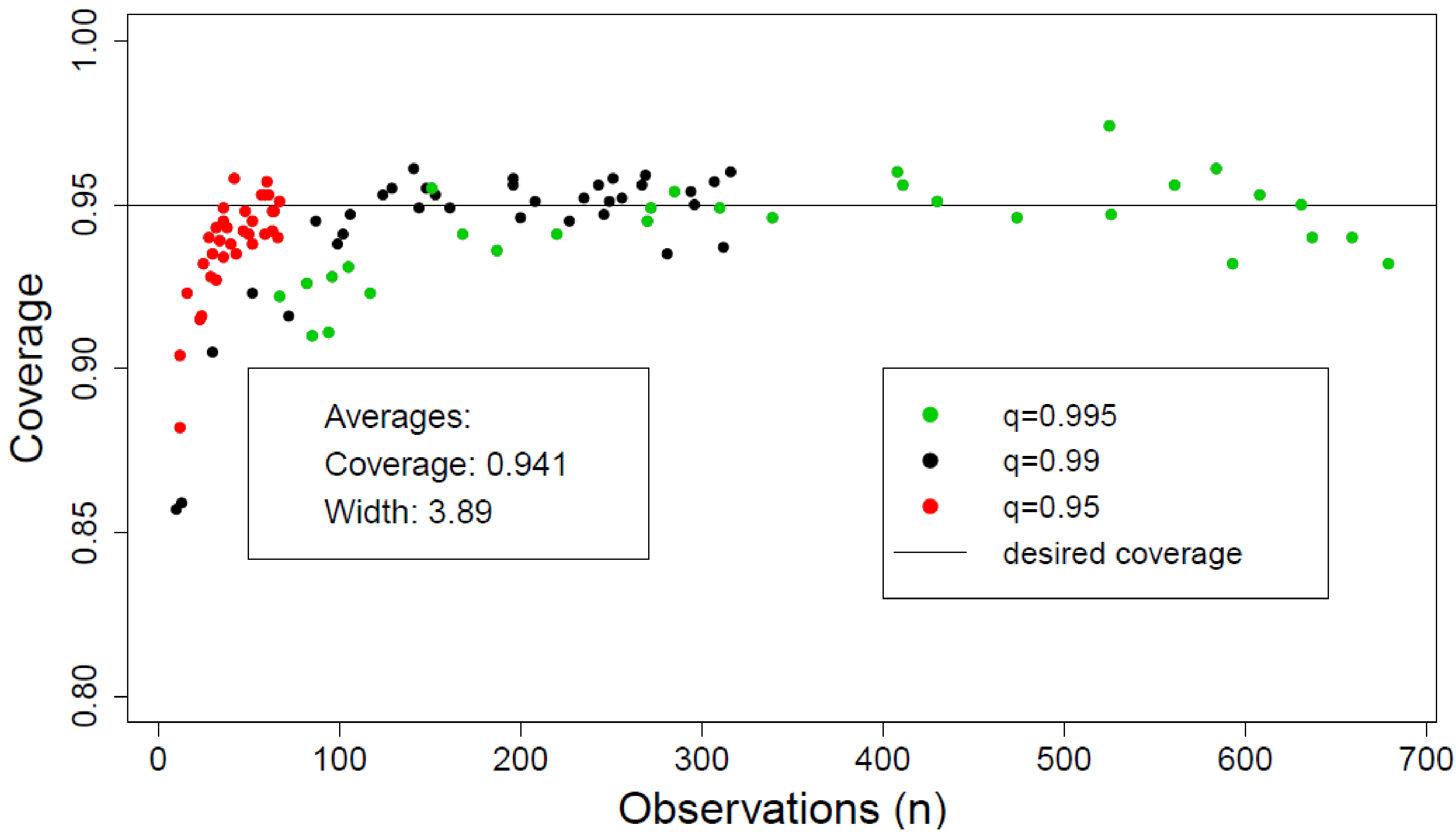

Figure: coverage rates based on 1000 samples for exponentially distributed data

Problems and Solutions

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- The required value of c is not depended on the distribution of the Data
- However, a vast improvement compared to the existing methods is achieved.
  - Drop in quality for data with very light/ heavy tails

- To further improve the method, a semi parametric approach is proposed:
  - Assume a distribution of the data and develop a model for the extension parameter using that distribution
  - Example: Exponential distribution

Modelling different distributions

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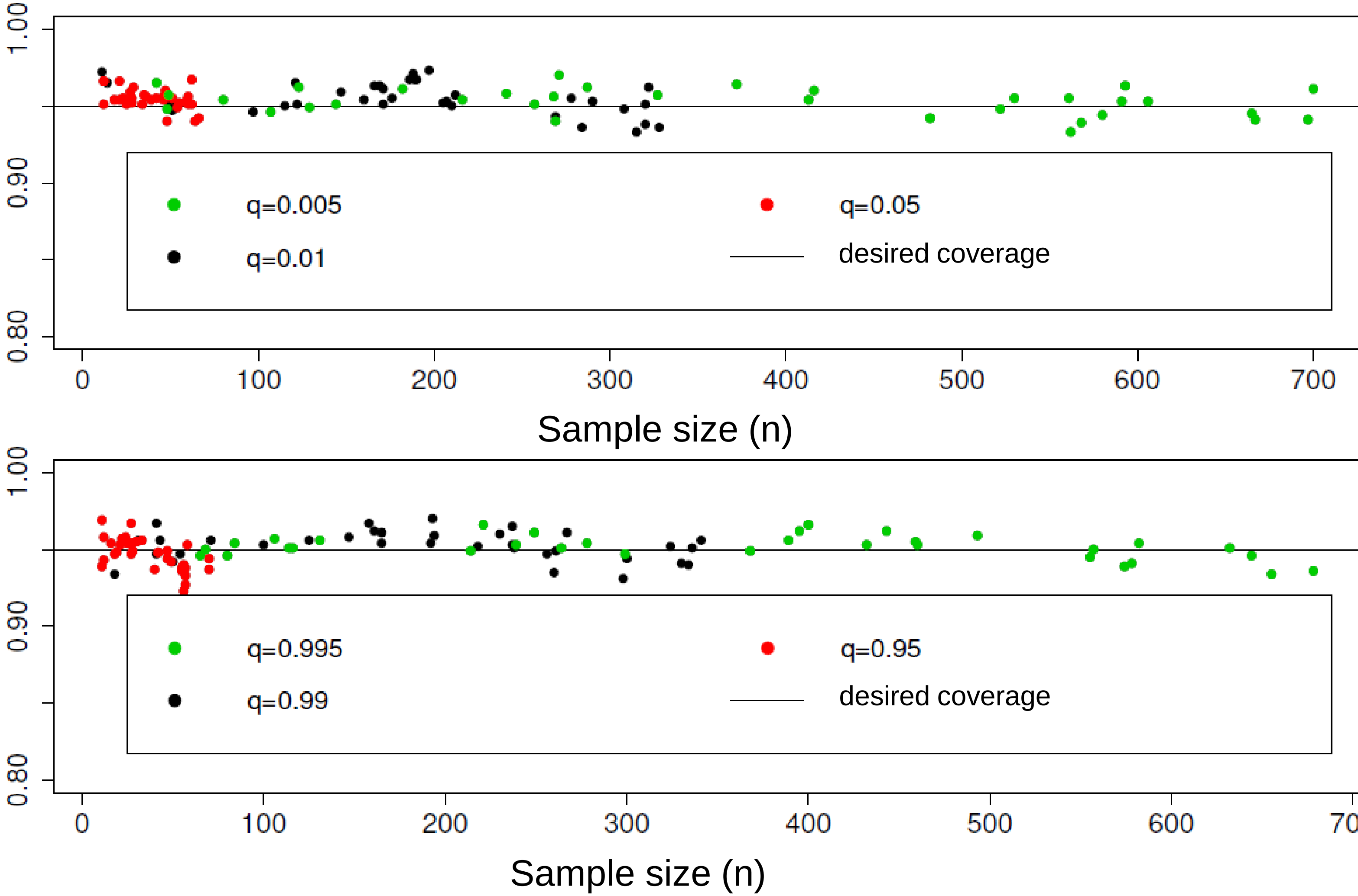

Figure: coverage rates based on 1000 samples for exponentially distributed data

Implementation

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Demonstration in JMP

Implementation: Summary

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- Straight forward programming of the functions and models
- Difficulty: Finding an algorithm for the borders of the CI
  - Minimize function: Fast but unstable in some situations
  - Simple self-made algorithm: slower but stable
- Development of a simple JMP application for user friendliness.